Convergence in Mean (L^2 Convergence) of Fourier Series

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Contents

• Euclidean Distance Between Discrete Signals

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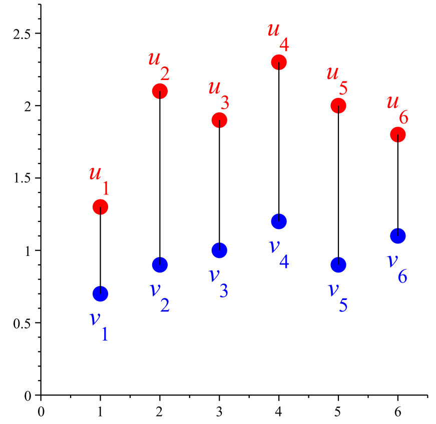

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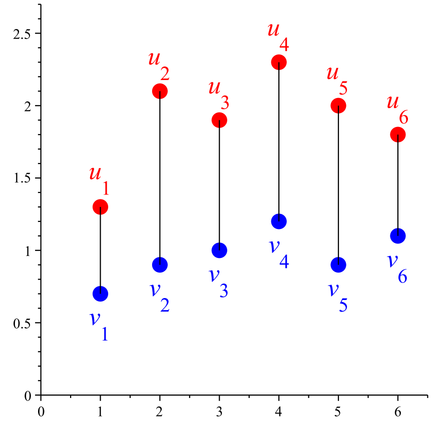
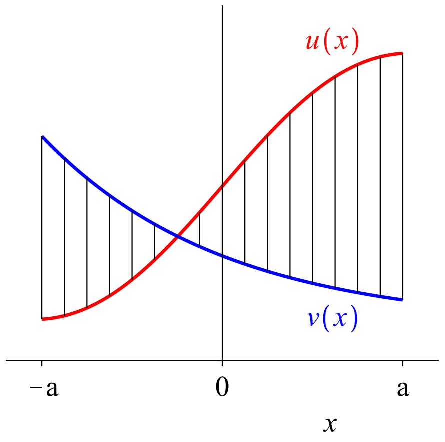


• L^2 Distance Between Functions

Given two functions $u(x)$ and $v(x)$ on $[-a, a]$,
the L^2 distance between them is

$$\left\{\int_{-a}^{a} [u(x) - v(x)]^2 dx\right\}^{1/2}.$$



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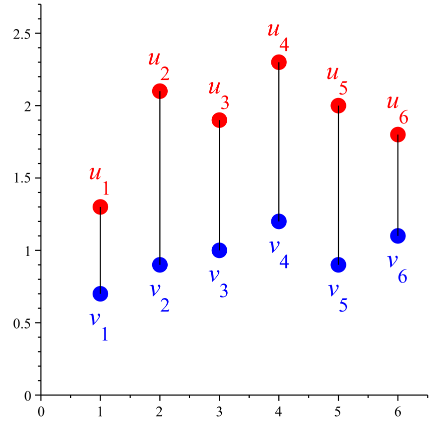

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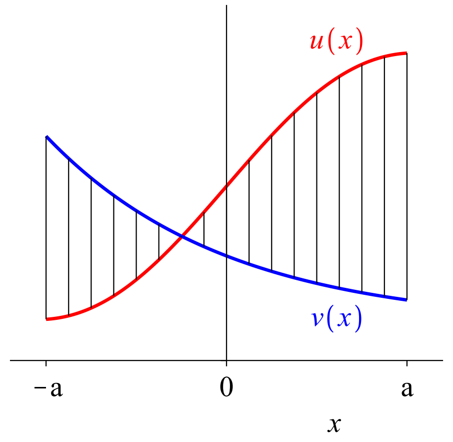


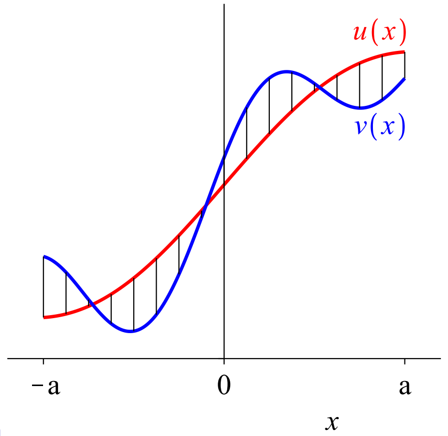
• Error in Mean

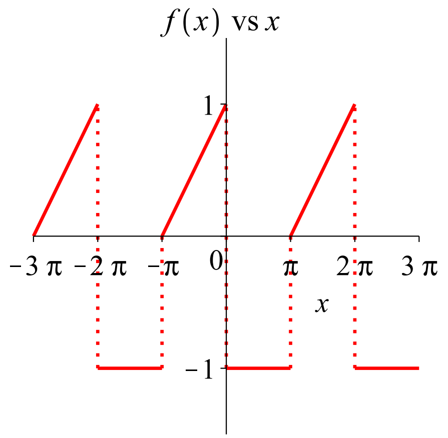
When a function $u(x)$ is approximated by $v(x)$,

$$(\text{Error in Mean}) = \left(L^2 \text{ distance}\right)^2.$$

In other words,

$$(\text{Error in Mean}) = \int_{-a}^{a} [u(x) - v(x)]^2 dx.$$

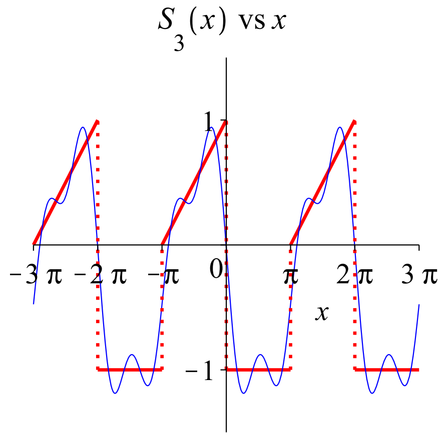




Example. $f(x)$ is 2π periodic,
$f(x) = 1 + x/\pi \ (-\pi \le x < 0)$, and $f(x) = -1 \ (0 \le x < \pi)$.

The Fourier seris of $f(x)$ is

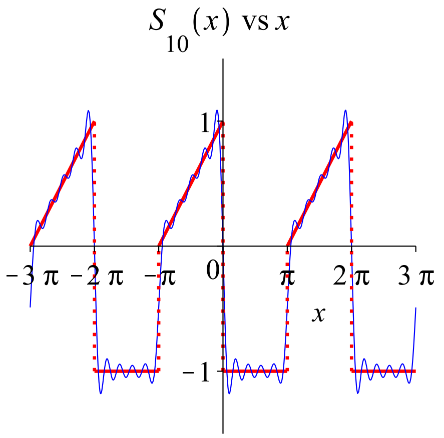
$$-\frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2+(-1)^n}{n\pi} \sin(nx) \right].$$



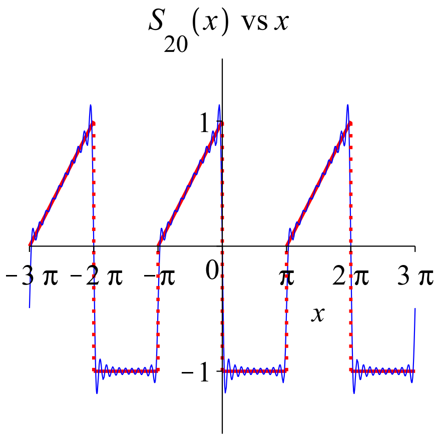
Approximate $f(x)$ by truncating the F-series at $N = 3$:

$$S_3(x) = -\frac{1}{4} + \sum_{n=1}^{3} \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2+(-1)^n}{n\pi} \sin(nx) \right].$$

$$(\text{Error in Mean}) = \int_{-\pi}^{\pi} [f(x) - S_3(x)]^2 \approx 0.4028159855$$



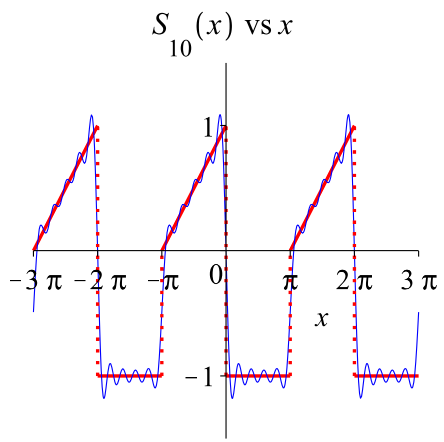
(Error in Mean) ≈ 0.1572187764



(Error in Mean) ≈ 0.07913602023

- Error in Mean decreases with N.
- Error in Mean $\to 0$, as $N \to \infty$.

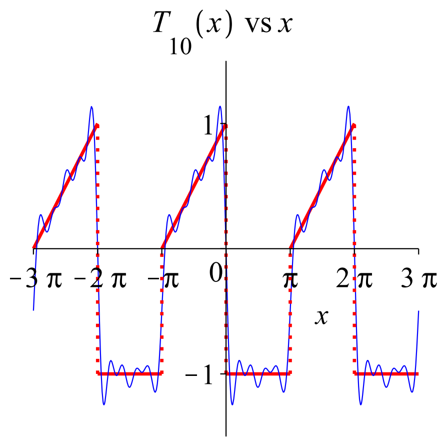
Can we choose other coefficients, to get better approximation?



For $N = 10$, approximate $f(x)$ by $S_{10}(x)$:

$$S_{10}(x) = -\frac{1}{4} + \sum_{n=1}^{10} \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2+(-1)^n}{n\pi} \sin(nx) \right].$$

(The Error in Mean of S_{10}) ≈ 0.1572187764



If we replace some Fourier coefficients by, say,

$a_3 = \frac{1}{50}, b_5 = -\frac{1}{4}, a_9 = \frac{1}{100},$ (just my random choices)

to form a new trig polynomial $T_{10}(x)$,

(the Error in Mean of T_{10}) ≈ 0.1683563939.

- Whatever coefficients you try, you can never beat S_{10}.
- Fourier coefficients are our best choices, in minimizing the error in mean.
- $S_{10}(x)$ is the best L^2 approx of $f(x)$, among all trig polynomials of degree 10.

Assumptions: $f(x)$ is $2a$ periodic and $\int_{-a}^{a} f(x)^2 dx < \infty$.

Let a_0, a_n, b_n be the Fourier coefficients of $f(x)$.

Let $S_N(x) = a_0 + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right)\right]$ (the truncated Fourier series of degree N)

Theorem (Best L^2 approximation)

$S_N(x)$ is the best L^2 approx of $f(x)$, among all trig polynomials of degree N.

More precisely, for any trig polynomial $T_N(x)$ of degree N,

$$(\text{the error in mean of } S_N) \leq (\text{the error in mean of } T_N).$$

Theorem (Convergence in mean. L^2 convergence.)

The error in mean of S_N decays to 0, as $N \to \infty$.

In other words, $S_N(x)$ converges to $f(x)$ in mean, as $N \to \infty$.

Formula (Parseval's equality)

$$\int_{-a}^{a} f(x)^2 dx = a\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$$

Formula (Error in mean of S_N)

$$\begin{aligned}(\text{The error in mean of } S_N) &= \int_{-a}^{a} f(x)^2 dx - a\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right] \\ &= a\left[\sum_{n=N+1}^{\infty}(a_n^2 + b_n^2)\right].\end{aligned}$$