Intro to Fourier Integral

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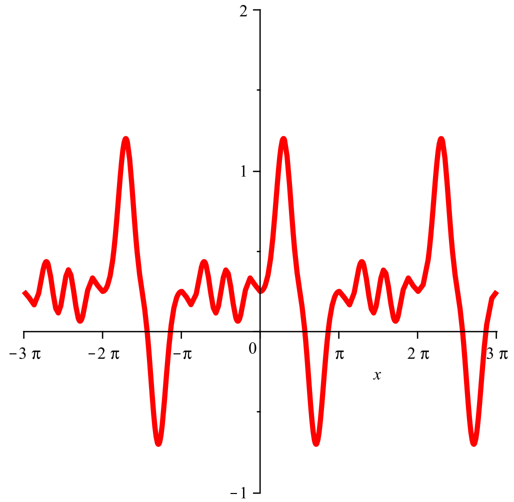
Examples:

- What is Fourier integral?
- Fourier integral formula
- An informal derivation of Fourier integral formula
- Examples

What is Fourier series?

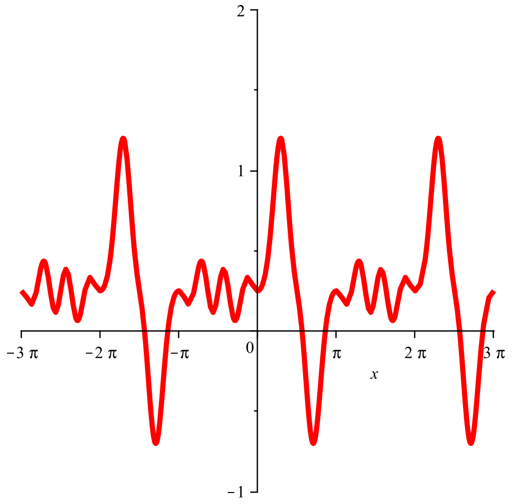


$f(x)$: $2a$-periodic function

$\Rightarrow$ Fourier series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right) + b_n \sin\left(\frac{n\pi}{a}x\right)$$

What is Fourier series?



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$\Rightarrow$ Fourier series:

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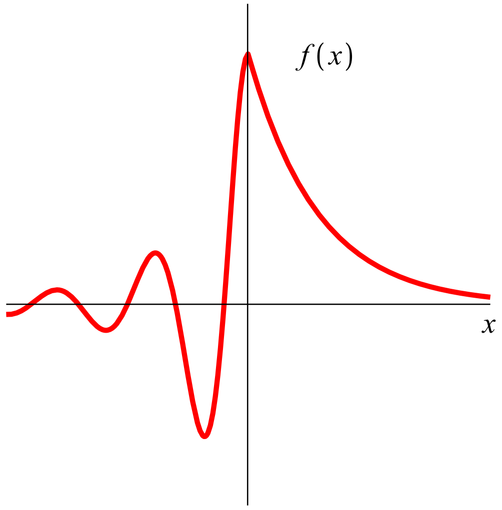
$f(x)$ is given by a "linear combin." of $\cos(\lambda x)$ & $\sin(\lambda x)$ with a discrete sequence of frequencies:

$$\lambda = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, \frac{4\pi}{a}, \cdots$$

What is Fourier integral?



$f(x)$: decaying (non-periodic) function

$\Rightarrow$ $f(x)$ is given by "linear combin." of $\cos(\lambda x)$ & $\sin(\lambda x)$

with a continuous interval of frequencies:

$$\text{all real numbers } \lambda \geq 0.$$

Fourier integral:

$$f(x) \sim \int_0^\infty \Big\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big\} d\lambda.$$

Fourier integral formula

If $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then

$$f(x) \sim \int_0^{\infty} \Big\{ A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x) \Big\} d\lambda,$$

where

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx, \qquad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\lambda x) dx.$$

Fourier integral formula

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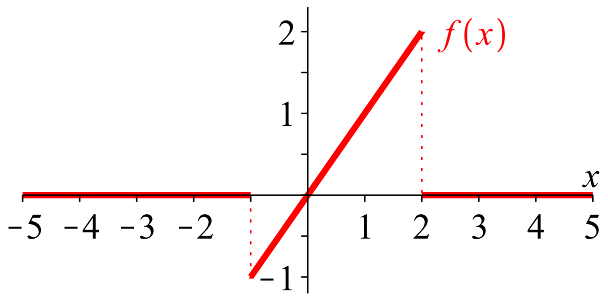
Convergence Theorem

If $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ and $f(x)$ is sectionally smooth, then

$$\int_0^{\infty} \Big\{A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\Big\}d\lambda = \frac{f(x+) + f(x-)}{2}.$$

Example 1.

$$f(x) = \begin{cases} x & \text{for } -1 < x < 2 \\ 0 & \text{everywhere else} \end{cases}$$



$$f(x) \sim \int_0^\infty \Big\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big\} d\lambda,$$

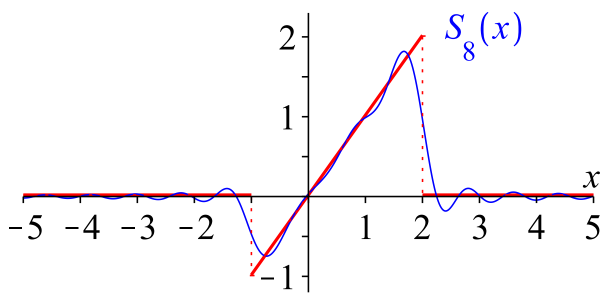
where

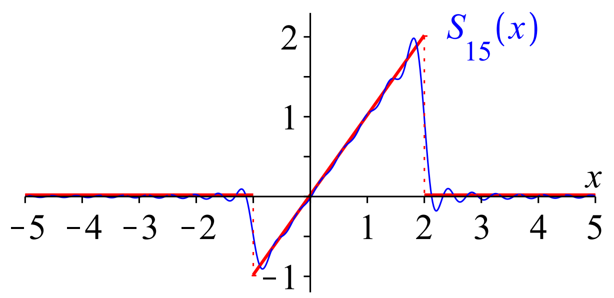
$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx = \frac{1}{\pi}\int_{-1}^{2} x\cos(\lambda x)dx = \frac{1}{\pi\lambda^2}[-\cos\lambda - \lambda\sin\lambda + \cos(2\lambda) + 2\lambda\sin(2\lambda)],$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx = \frac{1}{\pi}\int_{-1}^{2} x\sin(\lambda x)dx = \frac{1}{\pi\lambda^2}[\sin\lambda - \lambda\cos\lambda + \sin(2\lambda) - 2\lambda\cos(2\lambda)].$$

The truncated Fourier integral $S_N(x) = \int_0^N \Big\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big\} d\lambda,$

converges to $\frac{f(x+)+f(x-)}{2}$, as $N \to \infty$.

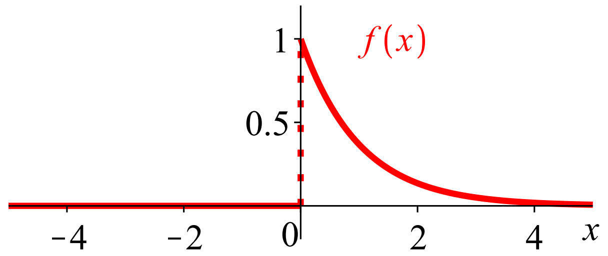




The Gibbs phenomenon occurs near the jumps $x = -1$ and $x = 2$.

Example 2.

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$



$$f(x) \sim \int_0^\infty \left\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \right\} d\lambda,$$

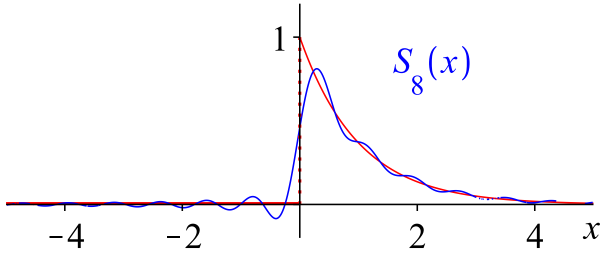
where

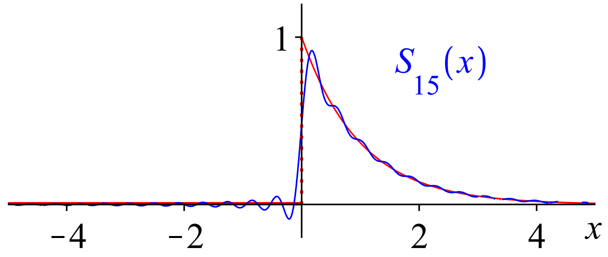
$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx = \frac{1}{\pi}\int_0^\infty e^{-x}\cos(\lambda x)dx = \frac{1}{\pi(1+\lambda^2)},$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx = \frac{1}{\pi}\int_0^\infty e^{-x}\sin(\lambda x)dx = \frac{\lambda}{\pi(1+\lambda^2)}.$$

The truncated Fourier integral $S_N(x) = \int_0^N \left\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \right\} d\lambda$,

converges to $\frac{f(x+)+f(x-)}{2}$, as $N \to \infty$.

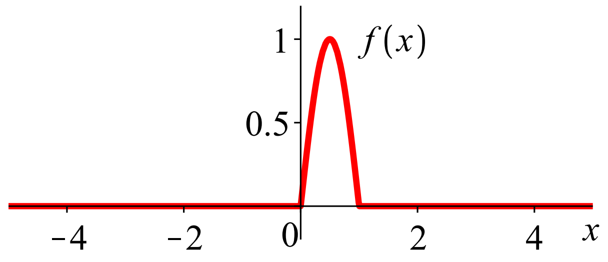




The Gibbs phenomenon occurs near the jump discontinuous point $x = 0$.

Example 3.

$$f(x) = \begin{cases} \sin(\pi x) & \text{for } 0 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$$



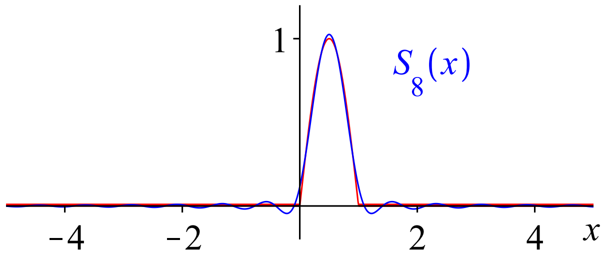
$$f(x) \sim \int_0^\infty \Big\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big\} d\lambda,$$

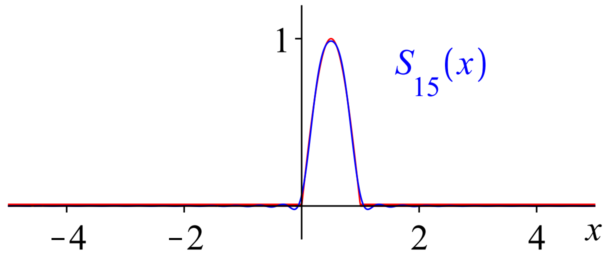
where

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx = \frac{1}{\pi}\int_0^1 \sin(\pi x)\cos(\lambda x)dx = \frac{1+\cos\lambda}{\pi^2-\lambda^2},$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx = \frac{1}{\pi}\int_0^1 \sin(\pi x)\sin(\lambda x)dx = \frac{\sin\lambda}{\pi^2-\lambda^2}.$$

The truncated Fourier integral $S_N(x) = \int_0^N \Big\{ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big\} d\lambda$, converges to $f(x)$, as $N \to \infty$.





The convergence is uniform, since $f(x)$ is continuous and sectionally smooth.