

## Solve Nonhomogeneous 1-D Heat Equation

### Example: Infinite Bar

**Objective:** Solve the initial value problem for a nonhomogeneous heat equation with zero initial condition:

$$(*) \begin{cases} u_t - ku_{xx} = p(x, t) & -\infty < x < \infty, t > 0, \\ u(x, 0) = f(x) & -\infty < x < \infty. \end{cases}$$

**Break into Two Simpler Problems:** The solution  $u(x, t)$  is the sum of  $u_1(x, t)$  and  $u_2(x, t)$ , which are solutions of the following problems respectively:

$$\begin{cases} u_{1t} - ku_{1xx} = 0 & -\infty < x < \infty, t > 0, \\ u_1(x, 0) = f(x) & -\infty < x < \infty. \end{cases}$$
$$\begin{cases} u_{2t} - ku_{2xx} = p(x, t) & -\infty < x < \infty, t > 0, \\ u_2(x, 0) = 0 & -\infty < x < \infty. \end{cases}$$

The solution  $u_1$  is obtained by using the heat kernel, while  $u_2$  is solved using Duhamel's principle.

**Solution Formula:** The solution of (\*) is given by

$$\begin{aligned} u(x, t) &= u_1(x, t) + u_2(x, t) \\ &= \int_{\xi=-\infty}^{\infty} G(x - \xi, t) f(\xi) d\xi + \int_{s=0}^t \int_{\xi=-\infty}^{\infty} G(x - \xi, t - s) p(\xi, s) d\xi ds \\ &= \int_{\xi=-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi + \int_{s=0}^t \int_{\xi=-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-\xi)^2}{4k(t-s)}} p(\xi, s) d\xi ds, \end{aligned}$$

for  $-\infty < x < \infty, t > 0$ .

### Example: Finite Bar

**Objective:** Solve the initial boundary value problem for a nonhomogeneous heat equation, with homogeneous boundary conditions and zero initial data:

$$(**) \begin{cases} u_t - ku_{xx} = p(x, t) & 0 < x < L, t > 0, \\ u(0, t) = T_0(t), u(L, t) = T_1(t) & t > 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L. \end{cases}$$

**Reduce the Boundary Conditions to Homogeneous:** Pick an arbitrary function  $v(x, t)$  satisfying the nonhomogeneous boundary conditions:

$$v(0, t) = T_0(t), v(L, t) = T_1(t) \quad t > 0.$$

For instance, one can take

$$v(x, t) = T_0(t) + \frac{x}{L}[T_1(t) - T_0(t)] \quad 0 \leq x \leq L, t \geq 0.$$

Consider  $w(x, t) = u(x, t) - v(x, t)$ , as the new unknown. The problem for  $w(x, t)$  becomes

$$(**)_w \begin{cases} w_t - kw_{xx} = q(x, t) & 0 < x < L, t > 0, \\ w(0, t) = 0, w(L, t) = 0 & t > 0, \\ w(x, 0) = g(x) & 0 \leq x \leq L, \end{cases}$$

where  $q(x, t)$  and  $g(x)$  are:

$$\begin{aligned} q(x, t) &= p(x, t) - v_t + kv_{xx} \\ &= p(x, t) - T_0'(t) - \frac{x}{L}[T_1'(t) - T_0'(t)], \\ g(x) &= f(x) - v(x, 0) \\ &= f(x) - T_0(0) - \frac{x}{L}[T_1(0) - T_0(0)]. \end{aligned}$$

**Break  $(**)_w$  into Two Simpler Problems:** The solution  $w(x, t)$  of  $(**)_w$  is the sum of  $w_1(x, t)$  and  $w_2(x, t)$ , which are solutions of the following problems respectively:

$$\begin{cases} w_{1t} - kw_{1xx} = 0 & 0 < x < L, t > 0, \\ w_1(0, t) = 0, w_1(L, t) = 0 & t > 0, \\ w_1(x, 0) = g(x) & 0 \leq x \leq L, \end{cases}$$

$$\begin{cases} w_{2t} - kw_{2xx} = q(x, t) & 0 < x < L, t > 0, \\ w_2(0, t) = 0, w_2(L, t) = 0 & t > 0, \\ w_2(x, 0) = 0 & 0 \leq x \leq L, \end{cases}$$

The solutions  $w_1$  and  $w_2$  can be obtained by using the Fourier series or the Green's function. In solving  $w_2$ , we apply Duhamel's principle.

**Solution Formula for (\*\*) in Fourier Series:** The original unknown  $u(x, t)$  is equal to  $v(x, t) + w(x, t) = v(x, t) + w_1(x, t) + w_2(x, t)$ . Thus,

$$u(x, t) = v(x, t) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) e^{-k(n\pi/L)^2 t} \\ + \sum_{n=1}^{\infty} \sin(n\pi x/L) \int_0^t B_n(s) e^{-k(n\pi/L)^2 (t-s)} ds,$$

for  $0 \leq x \leq L, t > 0$ , where  $b_n$  and  $B_n(s)$  are

$$b_n = \frac{2}{L} \int_0^L g(x) \sin(n\pi x/L) dx, \\ B_n(s) = \frac{2}{L} \int_0^L q(x, s) \sin(n\pi x/L) dx.$$

**Solution Formula for (\*\*) Using The Green's Function:** The original unknown  $u(x, t)$  is equal to  $v(x, t) + w(x, t) = v(x, t) + w_1(x, t) + w_2(x, t)$ . Thus,

$$u(x, t) = v(x, t) + \int_{\xi=0}^L G(x, t; \xi) g(\xi) d\xi + \int_{s=0}^t \int_{\xi=0}^L G(x, t-s; \xi) q(\xi, s) d\xi ds \\ = v(x, t) + \int_{\xi=0}^L \frac{1}{\sqrt{4\pi kt}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4kt)} - e^{-(x-2nL+\xi)^2/(4kt)} \right] g(\xi) d\xi ds \\ + \int_{s=0}^t \int_{\xi=0}^L \frac{1}{\sqrt{4\pi k(t-s)}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4k(t-s))} - e^{-(x-2nL+\xi)^2/(4k(t-s))} \right] q(\xi, s) d\xi ds$$

for  $0 \leq x \leq L, t > 0$ .

### Another Method: Use a Particular Solution

If one can find a particular solution  $v(x, t)$  of the nonhomogeneous problem, then a reduction to a homogeneous problem can be easily done by considering  $w(x, t) = u(x, t) - v(x, t)$ .

#### Example: a Finite Bar Problem

**Objective:** Solve the initial boundary value problem for a nonhomogeneous heat equation, with homogeneous boundary conditions and zero initial data:

$$(***) \begin{cases} u_t - ku_{xx} = p_0 & 0 < x < L, t > 0, \\ u(0, t) = T_0, u(L, t) = T_1 & t > 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L, \end{cases}$$

where  $p_0, T_0, T_1$  are constants.

**Time-Independent Solution:** One can easily find an equilibrium solution of  $(***)$ . The equations for time-independent solution  $v(x)$  of  $(***)$  are:

$$\begin{cases} -kv''(x) = p_0 & 0 < x < L \\ v(0) = T_0, v(L) = T_1. \end{cases}$$

Solving this we obtain

$$v(x) = \frac{p_0}{2k}x(L-x) + \frac{T_1 - T_0}{L}x + T_0 \quad 0 \leq x \leq L.$$

**The Reduced Homogeneous Problem for The Transient Temperature:** Consider the transient temperature

$$w(x, t) = u(x, t) - v(x).$$

The equations for  $w(x, t)$  form a homogeneous initial-boundary value problem:

$$(***)_w \begin{cases} w_t - kw_{xx} = 0 & 0 < x < L, t > 0, \\ w(0, t) = 0, w(L, t) = 0 & t > 0, \\ w(x, 0) = f(x) - v(x) & 0 \leq x \leq L, \end{cases}$$

which we know how to solve. Solving this we obtain the transient temperature  $w(x, t)$ . Finally the real temperature  $u(x, t)$  is given by

$$u(x, t) = v(x) + w(x, t).$$

## EXERCISES

- [1] Solve the above problem on infinite bar (\*) when  $p(x, t) = \delta(x - ct)$  and  $f(x) = 3\delta(x - 2) - \delta(x - 5)$ .
- [2] Solve the above problem on finite bar (\*\*) when  $p(x, t) = 0, T_0(t) = \sin t, T_1(t) = 0, f(x) = 0$ .
- [3] In the above problem on finite bar (\*\*\*) , let  $p(x, t) = 0, T_0(t) = 1, T_1(t) = 0, f(x) = 0$ . Find the time-independent solution  $v(x)$ , transient temperature  $w(x, t)$ , and temperature  $u(x, t)$ .
- [4] Find the solution formula for

$$\begin{cases} u_t - ku_{xx} = p(x, t) & 0 < x < L, t > 0, \\ u_x(0, t) = F_0(t), u_x(L, t) = F_1(t) & t > 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L. \end{cases}$$

- [5] Find the solution formula for

$$\begin{cases} u_t - ku_{xx} = p(x, t) & x > 0, t > 0, \\ u(0, t) = T_0(t) & t > 0, \\ u(x, 0) = f(x) & x \geq 0. \end{cases}$$

- [6] Show that a solution of

$$\begin{cases} u_t - ku_{xx} = 0 & x > 0, t > 0, \\ u(0, t) = T_0(t) & t > 0, \\ u(x, 0) = 0 & x \geq 0, \end{cases}$$

is given by

$$u(x, t) = \int_0^t \frac{x}{\sqrt{4\pi k(t-s)^3}} e^{-\frac{x^2}{4k(t-s)}} T_0(s) ds \quad x > 0, t > 0.$$

(Hint: Use the previous exercise.)

(See next page for the answers)

## ANSWERS

$$[1] \quad u(x, t) = \frac{3}{\sqrt{4\pi kt}} e^{-\frac{(x-2)^2}{4kt}} - \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-5)^2}{4kt}} + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-cs)^2}{4k(t-s)}} ds,$$

$$[2] \quad u(x, t) = (1 - x/L) \sin t + \sum_{n=1}^{\infty} \frac{2}{n\pi [k^2(n\pi/L)^4 + 1]} \left[ -\sin t - k(n\pi/L)^2 \cos t + k(n\pi/L)^2 e^{-k(n\pi/L)^2 t} \right] \sin \left( \frac{n\pi x}{L} \right)$$

$$[3] \quad v(x) = 1 - x/L,$$

$$w(x, t) = \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin \left( \frac{n\pi x}{L} \right) e^{-k(n\pi/L)^2 t},$$

$$u(x, t) = 1 - x/L + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin \left( \frac{n\pi x}{L} \right) e^{-k(n\pi/L)^2 t}$$

[4] The solution using Fourier series is

$$\begin{aligned} u(x, t) &= F_0(t)x + [F_1(t) - F_0(t)] \frac{x^2}{2L} \\ &\quad + a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) e^{-k(n\pi/L)^2 t} \\ &\quad + \int_0^t A_0(s) ds + \sum_{n=1}^{\infty} \cos(n\pi x/L) \int_0^t A_n(s) e^{-k(n\pi/L)^2 (t-s)} ds, \end{aligned}$$

for  $0 \leq x \leq L, t > 0$ , where  $a_n$  and  $A_n(s)$  are

$$a_0 = \frac{1}{L} \int_0^L g(x) dx, \quad a_n = \frac{2}{L} \int_0^L g(x) \cos(n\pi x/L) dx,$$

$$\text{with } g(x) = f(x) - F_0(0)x - [F_1(0) - F_0(0)] \frac{x^2}{2L},$$

$$A_0(s) = \frac{1}{L} \int_0^L q(x, s) dx, \quad A_n(s) = \frac{2}{L} \int_0^L q(x, s) \cos(n\pi x/L) dx,$$

$$\text{with } q(x, t) = p(x, t) - F'_0(t)x - [F'_1(t) - F'_0(t)] \frac{x^2}{2L} + \frac{k}{L} [F_1(t) - F_0(t)].$$

The solution using the Green's function is

$$\begin{aligned} u(x, t) &= F_0(t)x + [F_1(t) - F_0(t)] \frac{x^2}{2L} \\ &\quad + \int_{\xi=0}^L \frac{1}{\sqrt{4\pi kt}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4kt)} + e^{-(x-2nL+\xi)^2/(4kt)} \right] g(\xi) d\xi ds \\ &\quad + \int_{s=0}^t \int_{\xi=0}^L \frac{1}{\sqrt{4\pi k(t-s)}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4k(t-s))} + e^{-(x-2nL+\xi)^2/(4k(t-s))} \right] q(\xi, s) d\xi ds \end{aligned}$$

for  $0 \leq x \leq L, t > 0$ .

[5]

$$\begin{aligned}
u(x, t) &= T_0(t) + \int_{\xi=0}^{\infty} \frac{1}{\sqrt{4\pi kt}} \left[ e^{-\frac{(x-\xi)^2}{4kt}} - e^{-\frac{(x+\xi)^2}{4kt}} \right] \{f(\xi) - T_0(0)\} d\xi \\
&\quad + \int_{s=0}^t \int_{\xi=0}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left[ e^{-\frac{(x-\xi)^2}{4k(t-s)}} - e^{-\frac{(x+\xi)^2}{4k(t-s)}} \right] \{p(\xi, s) - T_0'(s)\} d\xi ds
\end{aligned}$$

[6] From the answer to the previous exercise,

$$\begin{aligned}
u(x, t) &= T_0(t) + \int_{\xi=0}^{\infty} G(x, t; \xi) \{ -T_0(0) \} d\xi + \int_{s=0}^t \int_{\xi=0}^{\infty} G(x, t-s; \xi) \{ -T_0'(s) \} d\xi ds \\
&= T_0(t) - T_0(0) - \int_{\xi=0}^{\infty} \left[ \int_{s=0}^t G(x, t-s; \xi) T_0'(s) ds \right] d\xi \\
&= T_0(t) - T_0(0) - \int_{\xi=0}^{\infty} \left[ G(x, t-s; \xi) T_0(s) \right]_{s=0}^{s=t} d\xi \\
&\quad + \int_{\xi=0}^{\infty} \int_{s=0}^t \frac{\partial}{\partial s} (G(x, t-s; \xi)) T_0(s) ds d\xi \\
&= T_0(t) - T_0(0) - \int_{\xi=0}^{\infty} \delta(x-\xi) T_0(t) d\xi + \int_{\xi=0}^{\infty} G(x, t; \xi) T_0(0) d\xi \\
&\quad + \int_{\xi=0}^{\infty} \int_{s=0}^t -\frac{\partial G}{\partial t}(x, t-s; \xi) T_0(s) ds d\xi \\
&= T_0(t) - T_0(0) - T_0(t) + T_0(0) \\
&\quad - \int_{s=0}^t \left[ \int_{\xi=0}^{\infty} k \frac{\partial^2 G}{\partial \xi^2}(x, t-s; \xi) T_0(s) d\xi \right] ds \\
&= - \int_{s=0}^t \left[ k \frac{\partial G}{\partial \xi}(x, t-s; \xi) T_0(s) \right]_{\xi=0}^{\xi=\infty} ds \\
&= \int_{s=0}^t k \frac{\partial G}{\partial \xi}(x, t-s; 0) T_0(s) ds \\
&= \int_{s=0}^t k \frac{1}{\sqrt{4\pi k(t-s)}} \frac{2x}{2k(t-s)} e^{-\frac{x^2}{4k(t-s)}} T_0(s) ds \\
&= \int_{s=0}^t \frac{x}{\sqrt{4\pi k(t-s)^3}} e^{-\frac{x^2}{4k(t-s)}} T_0(s) ds.
\end{aligned}$$