Nonhomogeneous 1-D Heat Equation

Duhamel's Principle on Infinite Bar

Objective: Solve the initial value problem for a nonhomogeneous heat equation with zero initial condition:

$$(*)\begin{cases} u_t - ku_{xx} = p(x,t) & -\infty < x < \infty, t > 0, \\ u(x,0) = 0 & -\infty < x < \infty. \end{cases}$$

An Auxiliary Problem: For every fixed $s > 0$, consider a homogeneous heat equation for $t > s$, with $p(x,s)$ as the intial data at time $t = s$:

$$\begin{cases} u_t(x,t;s) - ku_{xx}(x,t;s) = 0 & -\infty < x < \infty, t > s, \\ u(x,s;s) = p(x,s) & -\infty < x < \infty. \end{cases}$$

The solution $u(x,t;s)$ of this auxiliary problem (parametrized by s) is given by

$$u(x,t;s) = \int_{-\infty}^{\infty} G(x-\xi, t-s)p(\xi,s)d\xi \qquad -\infty < x < \infty, t > s,$$

where $G(x,t)$ is the heat kernel: $G(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}$.

Duhamel's Principle: The solution of (*) is given by

$$u(x,t) = \int_0^t u(x,t;s)ds.$$

Thus, we have

$$\begin{aligned} u(x,t) &= \int_{s=0}^{t}\int_{\xi=-\infty}^{\infty} G(x-\xi,t-s)p(\xi,s)d\xi ds \\ &= \int_{s=0}^{t}\int_{\xi=-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}}e^{-\frac{(x-\xi)^2}{4k(t-s)}}p(\xi,s)d\xi ds \qquad -\infty < x < \infty, t > 0. \end{aligned}$$

Duhamel's Principle on Finite Bar

Objective: Solve the initial boundary value problem for a nonhomogeneous heat equation, with homogeneous boundary conditions and zero initial data:

$$(**)\begin{cases} u_t - ku_{xx} = p(x,t) & 0 < x < L, t > 0, \\ u(0,t) = 0, u(L,t) = 0 & t > 0, \\ u(x,0) = 0 & 0 \le x \le L. \end{cases}$$

An Auxiliary Problem: For every fixed $s > 0$, consider a homogeneous heat equation for $t > s$, with the same homoegeneous boundary conditions and with $p(x,s)$ as the intial data at time $t = s$:

$$\begin{cases} u_t(x,t;s) - ku_{xx}(x,t;s) = 0 & 0 < x < L, t > s, \\ u(0,t;s) = 0, u(L,t;s) = 0, & t > s, \\ u(x,s;s) = p(x,s) & 0 \le x \le L. \end{cases}$$

The solution $u(x,t;s)$ of this auxiliary problem (parametrized by s) is given by

$$u(x,t;s) = \int_0^L G(x, t-s; \xi) p(\xi, s) d\xi \qquad 0 \le x \le L, t > s,$$

where $G(x,t;\xi)$ is the Green's function:

$$\begin{aligned} G(x,t;\xi) &= \sum_{n=-\infty}^{\infty} [G(x - 2nL - \xi, t) - G(x - 2nL + \xi, t)] \\ &= \frac{1}{\sqrt{4\pi kt}} \sum_{n=-\infty}^{\infty} \left[e^{-(x-2nL-\xi)^2/(4kt)} - e^{-(x-2nL+\xi)^2/(4kt)} \right]. \end{aligned}$$

Duhamel's Principle: The solution of $(**)$ is given by

$$u(x,t) = \int_0^t u(x,t;s) ds.$$

Thus, we have

$$\begin{aligned} u(x,t) &= \int_{s=0}^t \int_{\xi=0}^L G(x, t-s; \xi) p(\xi, s) d\xi ds \\ &= \int_{s=0}^t \int_{\xi=0}^L \frac{1}{\sqrt{4\pi k(t-s)}} \sum_{n=-\infty}^{\infty} \left[e^{-(x-2nL-\xi)^2/(4k(t-s))} - e^{-(x-2nL+\xi)^2/(4k(t-s))} \right] p(\xi,s) d\xi ds, \end{aligned}$$

for $0 \le x \le L, t > 0$.

Another expression using Fourier Series: We can also use Fourier series to solve the above auxiliary problem:

$$u(x,t;s) = \sum_{n=1}^{\infty} b_n(s) \sin(n\pi x/L) e^{-k(n\pi/L)^2(t-s)} \qquad 0 \le x \le L, t > s,$$

where the coefficients $b_n(s)$ are determined by the initial dada of the auxiliary problem:

$$b_n(s) = \frac{2}{L}\int_0^L p(x,s)\sin(n\pi x/L)dx.$$

Then, by Duhamel's Principle, the solution of $(**)$ is given by

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t;s)ds \\ &= \sum_{n=1}^{\infty} \sin(n\pi x/L)\int_0^t b_n(s)e^{-k(n\pi/L)^2(t-s)}ds, \end{aligned}$$

for $0 \leq x \leq L, t > 0$.

EXERCISES

[1] Solve the above problem on infinite bar (*) when $p(x,t) = \delta(x - ct)$, with $c =$ constant.

[2] Solve the above problem on finite bar ($**$) when $p(x,t) = x \sin t$, using the Fourier series expression.

[3] Find the solution formula for

$$\begin{cases} u_t - k u_{xx} = p(x,t) & 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0 & t > 0, \\ u(x,0) = 0 & 0 \le x \le L. \end{cases}$$

[4] Find the solution formula for

$$\begin{cases} u_t - k u_{xx} = p(x,t) & x > 0, t > 0, \\ u(0,t) = 0 & t > 0, \\ u(x,0) = 0 & x \ge 0. \end{cases}$$

(See next page for the answers)

ANSWERS

[1] $u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-cs)^2}{4k(t-s)}} ds$, or, equivalently, $u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k\tau}} e^{-\frac{(x-ct+c\tau)^2}{4k\tau}} d\tau$

[2] $u(x,t) = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi\left[k^2(n\pi/L)^4+1\right]} \left(-\cos t + k(n\pi/L)^2 \sin t + e^{-k(n\pi/L)^2 t}\right) \sin\left(\frac{n\pi x}{L}\right)$

[3] $u(x,t) = \int_{s=0}^t \int_{\xi=0}^L \frac{1}{\sqrt{4\pi k(t-s)}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(x-2nL-\xi)^2}{4k(t-s)}} + e^{-\frac{(x-2nL+\xi)^2}{4k(t-s)}}\right] p(\xi,s)d\xi ds,$

or, equivalently,

$$u(x,t) = \int_0^t a_0(s)ds + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \int_0^t a_n(s) e^{-k\left(\frac{n\pi}{L}\right)^2(t-s)} ds,$$

where $a_0(s) = \frac{1}{L}\int_0^L p(x,s)dx, \quad a_n(s) = \frac{2}{L}\int_0^L p(x,s)\cos\left(\frac{n\pi x}{L}\right)dx.$

[4] $u(x,t) = \int_{s=0}^t \int_{\xi=0}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left[e^{-\frac{(x-\xi)^2}{4k(t-s)}} - e^{-\frac{(x+\xi)^2}{4k(t-s)}}\right] p(\xi,s)d\xi ds$