

What are Partial Differential Equations (PDEs)?

A PDE is an equation involving an unknown function u of multi-variables and its partial derivatives.

PDEs are usually employed to describe continuous processes in physical and engineering systems.

PDE Examples

Linear Equations

- 1-D HEAT EQ: $u_t = k u_{xx}$
- 1-D WAVE EQ: $u_{tt} = c^2 u_{xx}$
- 2-D LAPLACE EQ: $\Delta u = u_{xx} + u_{yy} = 0$
- 1-D LINEAR TRANSPORT EQ: $u_t + b u_x = 0$
- POISSON EQ: $\Delta u = f$
- KLEIN-GORDON EQ: $u_{tt} - c^2 \Delta u + m^2 u = 0$
- SCHRÖDINGER EQ: $i \hbar u_t = -\frac{\hbar^2}{2m} \Delta u + V(x)u$
- BIHARMONIC EQ: $\Delta^2 u = \Delta(\Delta u) = 0$

Nonlinear Equations

- BERGER'S EQ: $u_t + u u_x = 0$
- REACTION DIFFUSION EQ: $u_t = k \Delta u + f(u)$
- KdV EQ: $u_t + u u_x + u_{xxx} = 0$
- MINIMAL SURFACE EQ: $\operatorname{div} \left[\frac{\operatorname{grad} u}{\sqrt{1 + |\operatorname{grad} u|^2}} \right] = 0$

Linear Systems

- 3-D MAXWELL'S EQS: $\varepsilon \mathbf{E}_t - \operatorname{curl} \mathbf{H} = 0, \mu \mathbf{H}_t + \operatorname{curl} \mathbf{E} = 0, \operatorname{div} \mathbf{E} = 0, \operatorname{div} \mathbf{H} = 0$
- LINEAR ELASTICITY: $\rho \mathbf{u}_{tt} = \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad}(\operatorname{div} \mathbf{u}),$ where $\mathbf{u} = (u_1, u_2, u_3)$

Nonlinear Systems

- REACTION DIFFUSION SYSTEM: $\mathbf{u}_t = A \Delta \mathbf{u} + \mathbf{f}(\mathbf{u})$
- NAVIER-STOKES EQUATIONS: $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} = -\operatorname{grad} p, \operatorname{div} \mathbf{u} = 0$
- EULER EQUATIONS: $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \operatorname{div} \mathbf{u} = 0$

Generally speaking, PDEs are more difficult than ODEs. There are, times, however, a PDE problem can be reduced to an ODE problem under special assumptions. This helps a lot in constructing particular solutions of PDEs and sometimes even helps to solve the PDE problem completely. I will assume you are familiar with at least the following ODE topics:

- **Scalar separable equations:** $\frac{dy}{dx} = f(x)g(y)$.
- **First order scalar linear equations:** $\frac{dy}{dx} + a(x)y = b(x)$.
- **Linear ODEs and systems.**

We will focus on several particular types of PDEs (mostly only linear equations) that allow us to find explicit solution formulas.

First order PDEs. The basic technique will be the *method of characteristics*. The method consists of mainly two steps. First, by solving an auxiliary system of ODEs, we construct a family of curves, the so-called characteristics, emanating from the initial line/surface and sweeping out (at least an open region in) the space-time. Next, by tracing the behavior of the solution along every characteristics, we shall be able to find the solution.

Second order hyperbolic equations. A second order hyperbolic equation is not explicitly solvable in general. We will nevertheless investigate several particular cases which can be successfully treated by specially designed methods. The most important example in this class will be the 1-D wave equation, which is a classical model for small vibrations of string. We will discuss two methods. The first method of solving the 1-D wave equation is a special change of variables, the so-called D'Alembert's method. The fundamental solution will be discussed. Another important approach is "Separation of Variables". This will lead naturally to the use of Fourier series.

Second order parabolic equations. The heat equation belongs to this class. We will solve the initial value problem by using the fundamental solution. This combined with appropriate symmetric extensions can be used to solve some boundary value problems of the heat equation. Another method is to use "Separation of Variables". Fourier developed his theory of Fourier series for solving the one-dimensional heat conduction problem.

Second order elliptic equations. We will mainly consider Laplace's equation in this class. Again, the methods of "Fundamental Solutions" and "Separation of Variables" will be discussed. Some boundary value problems will be solved explicitly.