Shock Formation in Scalar First Order Nonlinear PDEs

Traffic Flow Model: Revisited

We considered first order nonlinear PDEs in last class . Let us see how a traffic flow model with somewhat reasonable assumptions can naturally lead to a first order nonlinear equation.

The problem is a continuum model of traffic flow along a straight road, which is the x-axis. For simplicity we assume this is a single lane highway without cars coming in from outside or cars getting off from the road; in other words, the nonhomogeneous term is zero.

We also assume that *the vehicle velocity is a function of the density* ρ:

$$V = V(\rho)$$

Then the flux of the traffic flow will be $F(\rho) = V(\rho)\rho$. The PDE becomes:

$$\boxed{\rho_t + (V(\rho)\rho)_x = 0.} \tag{0.1}$$

Or, equivalently,

$$\rho_t + (F(\rho))_x = \rho_t + F'(\rho)\rho_x = 0.$$

From the consideration of characteristics, the propagation speed of the traffic wave is equal to the coefficient of the space derivative term, that is,

traffic propagation speed $= F'(\rho) = V(\rho) + V'(\rho)\rho$. [1]

Example.

Consider

$$\rho_t + \rho\rho_x = 0 \quad x \in \mathbb{R}, t > 0,$$
$$\rho(x,0) = g(x) = \begin{cases} 1 & x \le 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x \ge 1. \end{cases}$$

STEP 1: Characteristic curves are given by

$$\begin{cases} t = s \\ x = \xi + g(\xi)s \\ U = g(\xi). \end{cases}$$

The solution remains constant along each characteristic line. The difficulty in this example is that some characteristic lines will intersect each other.

For instance, we choose first $\xi = -1$ and follow the corresponding characteristic line $t = s, x = -1 + s$, or, $x = t - 1$. We have $\rho(t-1, t) = g(-1) = 1$; in particular, $\rho(1, 2) = 1$.

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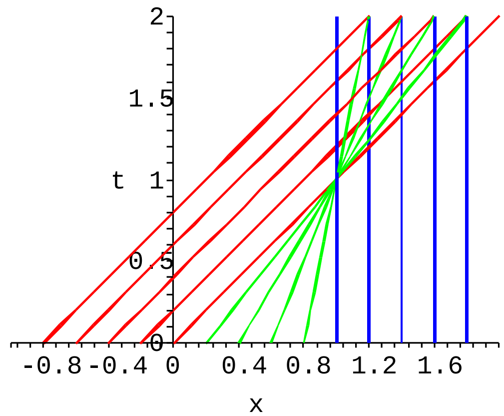
[1] In general, the traffic propagation speed $\neq$ the car velocity. For instance, assume that $V(\rho)$ is a decreasing function of ρ, $V'(\rho) < 0$; that means the cars move slower in heavier traffic. The traffic propagation speed $= V(\rho) + V'(\rho)\rho < V(\rho)$. If we are considering a traffic moving to the right, then we can observe the traffic waves propagating backward in the traffic.

On the other hand, we can also choose $\xi = 1$. The corresponding characteristic line is a vertical $x = 1$. So, the solution satisfies $\rho(1,t) = g(1) = 0$ for any t, contradicting the previous computation $\rho(1,2) = 1$. In other words, at the intersection point $(1,2)$ of the above two characteristic lines, the solution becomes multi-valued.

The first such intersection of characteristic lines occurs at $(x,t) = (1,1)$. This pathology means that the solution in the usual sense fails to exist after $t \geq 1$.

Characteristics_in_Shock_Formation



STEP 2: Now we solve inverse function of $t = s, x = \xi + g(\xi)s$. Because of the intersection issue explained above, this will only make sense for $0 \leq t < 1$.

For $x \leq t < 1$, we get

$$s = t, \xi = x - t.$$

For $0 \leq x \leq 1, x \geq t$,

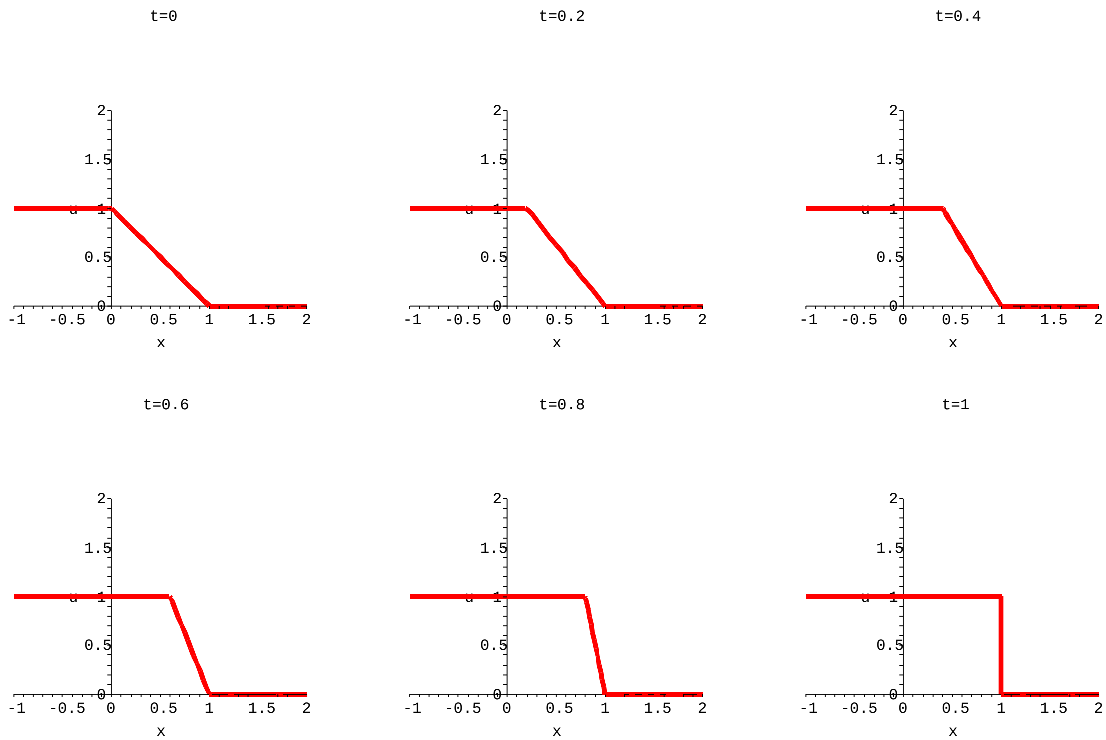
$$s = t, \xi = (x - t)/(1 - t).$$

For $x \geq 1, 0 \leq t < 1$,

$$s = t, \xi = x.$$

STEP 3: Now the solution for $x \in \mathbb{R}, 0 \leq t < 1$ is given by

$$\rho(x,t) = \begin{cases} 1 & x \leq t, 0 \leq t < 1, \\ (1-x)/(1-t) & t \leq x < 1, 0 \leq t < 1, \\ 0 & x \geq 1, 0 \leq t < 1. \end{cases}$$



We have seen in the above that the solution of nonlinear equations may develop discontinuities, i.e., shocks. What happens after the shock development? Roughly speaking, the shock persists (at least for a period of time) and will move in space. The velocity of shock propagation is determined by the mis-match of the solution at the discontinuity. The equation of motion for shocks is called the *Rankine-Hugoniot condition*.

A derivation of the Rankine-Hugoniot condition requires the notion of weak solutions.

A smooth function $\phi(x,t)$ is called a test function on the upper half plane $\{x \in \mathbb{R}, t > 0\}$ if $\phi = 0$ whenever t is small enough, t is large enough, or x is large enough. In other words, $\phi(x,t)$ is nonzero only in a bounded region that is away from the x-axis.

A function is called a *weak solution* of $\rho_t + (F(\rho))_x = 0$ for $x \in \mathbb{R}, t > 0$ if it satisfies the integral identity

$$\int_{t=0}^{\infty} \int_{x=-\infty}^{\infty} \left[\rho(x,t)\phi_t(x,t) + F(\rho(x,t))\phi_x(x,t)\right] dxdt = 0,$$

for any test function ϕ on the upper half plane $\{x \in \mathbb{R}, t > 0\}$.

The notion of weak solution is a reasonable definition because in order for a smooth function $\rho(x,t)$ to satisfy the PDE $\rho_t + (F(\rho))_x = 0$ in $\{x \in \mathbb{R}, t > 0\}$, the above integral identity is a sufficient and necessary condition. This can be shown by applying Green's theorem (the integral theorem relating the line integral to a double integral).

Furthermore, by using Green's Theorem, we can also give the following description concerning the shock movement in a weak solution.

Theorem. *Let $\rho(x,t)$ be a weak solution of $\rho_t + (F(\rho))_x = 0$ in $\{x \in \mathbb{R}, t > 0\}$. Assume that*

- Γ *is a smooth curve in the spacetime plane, which is parametrized by,* Γ*:* $x = a(t)$ *for* $t_1 \leq t \leq t_2$*;*
- $\rho(x,t)$ *is smooth away from the curve* Γ*;*
- $\rho(x,t)$ *has jump discontinuities along* Γ*; that is, the one-sided limits*

$$\rho_+(t) = \lim_{x \to a(t)+} \rho(x,t) \text{ and } \rho_-(t) = \lim_{x \to a(t)-} \rho(x,t)$$

both exist for every $t_1 \leq t \leq t_2$*, but the two limits are not equal.*

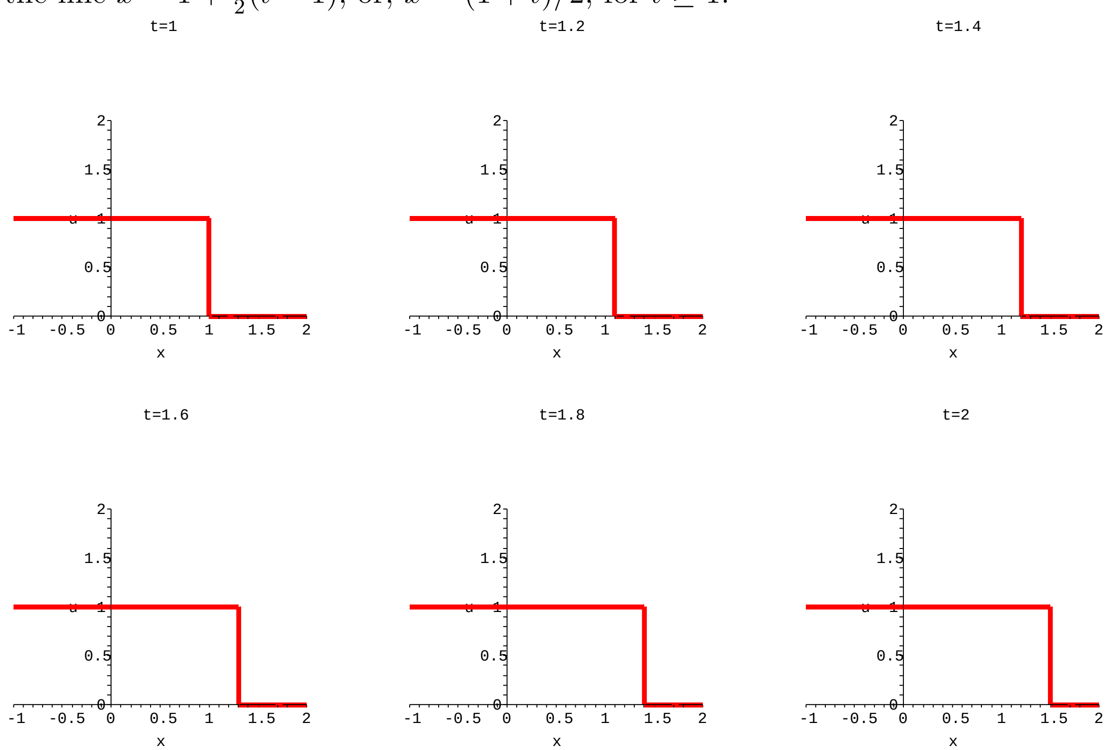
Then the curve Γ *must satisfy*

$$a'(t) = \frac{F(\rho_+(t)) - F(\rho_-(t))}{\rho_+(t) - \rho_-(t)}. \qquad \textbf{(Rankine-Hugoniot Condition)}$$

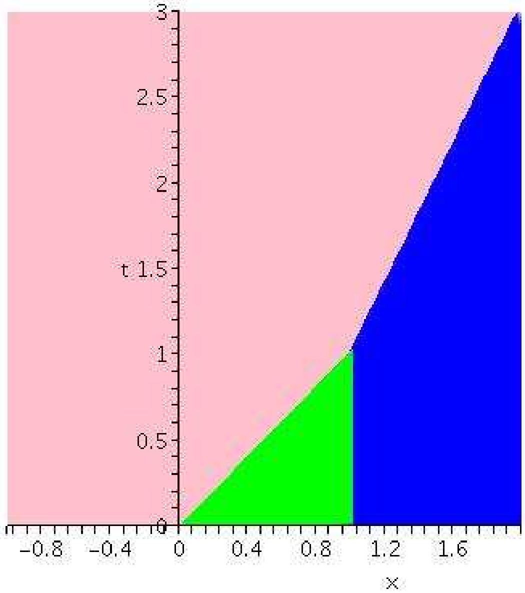
Going back to the Example, the Rankine-Hugoniot Condition helps to get a solution after the shock formation, a weak solution for all $t \geq 0$, including $t \geq 1$. In this particular exmaple, $F(\rho) = \frac{1}{2}\rho^2$, $\rho_+ = 0$, and $\rho_- = 1$. Hence,

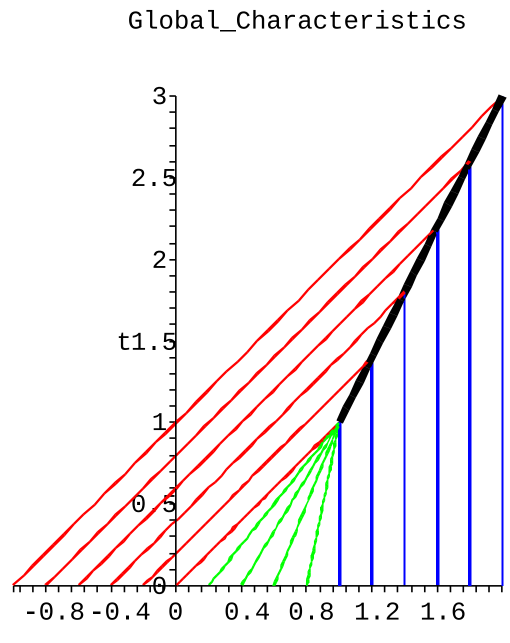
$$\text{the shock speed } = \frac{\frac{1}{2}\rho_+^2 - \frac{1}{2}\rho_-^2}{\rho_+ - \rho_-} = \frac{\rho_+ + \rho_-}{2} = \frac{1}{2}.$$

Therefore, a shock is formed at $(x,t) = (1,1)$; after that, it moves with constant speed $1/2$, along the line $x = 1 + \frac{1}{2}(t-1)$, or, $x = (1+t)/2$, for $t \geq 1$.



In conclusion, the solution $\rho(x,t)$ takes different forms in the three different regions in the spacetime plane.





In the figure on the right, we sketched characteristic lines globally in time (even after the shock formation moment). The black line in the figure represents the shock trajectory.

$$\rho(x,t) = \begin{cases} 1 & \text{for } (x \le t, 0 \le t < 1), \text{ or } (x < (1+t)/2, t \ge 1); \\ 0 & \text{for } (x \ge 1, 0 \le t < 1), \text{ or } (x > (1+t)/2, t \ge 1); \\ (1-x)/(1-t) & \text{for } t \le x \le 1, 0 \le t < 1. \end{cases}$$

Example 2.

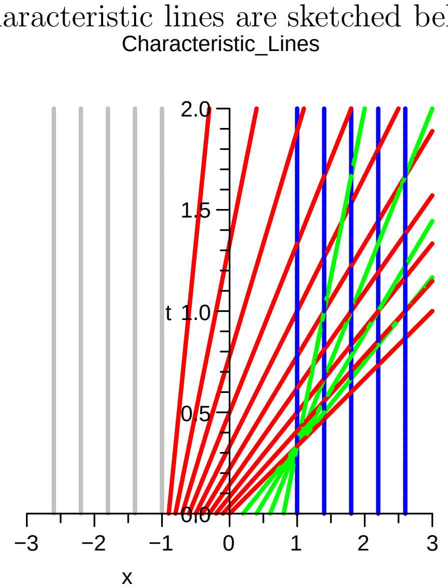
Let's consider another example. Solve

$$u_t + uu_x = 0 \qquad -\infty < x < \infty, t > 0,$$
$$u(x,0) = g(x) = \begin{cases} 3(1-|x|) & |x| \le 1, \\ 0 & |x| > 1. \end{cases}$$

Solution: Part (a): *Find the solution before the shock appears.*
Compute characteristics; that is, solve the following system of ODEs:

$$\frac{dt}{ds} = 1, t(0) = 0,$$
$$\frac{dx}{ds} = U, x(0) = \xi,$$
$$\frac{dU}{ds} = 0, U(0) = g(\xi).$$

Solve the first eq $\Rightarrow t = s$.
Solve the third eq $\Rightarrow U = constant = g(\xi)$.
Plug U in the second eq and solve $\Rightarrow x = \xi + g(\xi)t$.
The characteristic lines are sketched below:



The first intersection of the characteristic lines for $t > 0$ occurs at $t = 1/3$ and $x = 1$.

• The grey region is given by $x \le -1$. In the grey region, the characteristics are vertical lines. We have $t = s, x = \xi$ and hence $u(x,t) = g(\xi) = 0$.

• The red region is given by $-1 < x \le 3t$. Here,

$$t = s, x = \xi + 3(1+\xi)t \Rightarrow s = t, \xi = \frac{x-3t}{1+3t}.$$

The corresponding ξ is between -1 and 0. Hence,

$$u(x,t) = g(\xi) = 3(1+\xi) = 3\left(1 + \frac{x-3t}{1+3t}\right) = \frac{3(1+x)}{1+3t}.$$

• The green region is given by $3t < x < 1$. Here,

$$t = s, x = \xi + 3(1-\xi)t \Rightarrow s = t, \xi = \frac{x-3t}{1-3t}.$$

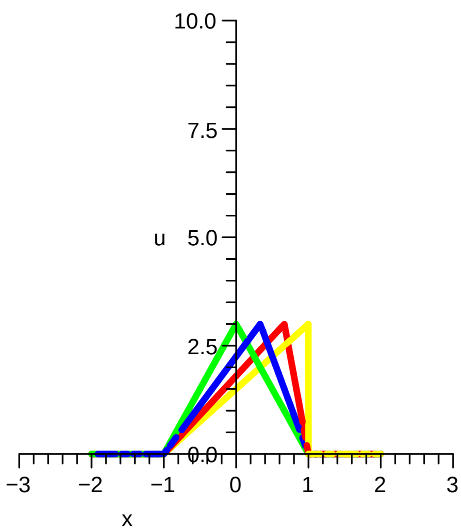
The corresponding ξ is in the interval $(0,1)$. Hence,

$$u(x,t) = g(\xi) = 3(1-\xi) = 3\left(1 - \frac{x-3t}{1-3t}\right) = \frac{3(1-x)}{1-3t}.$$

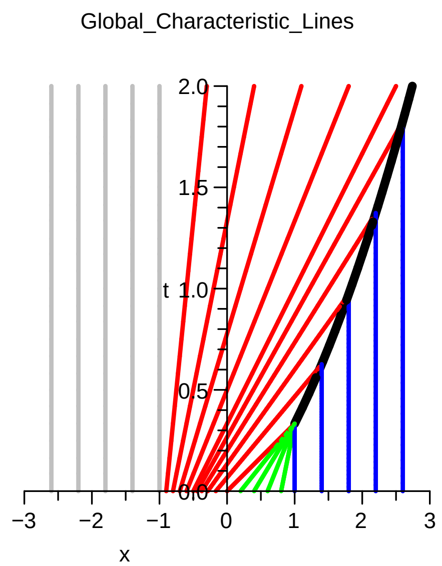
• The blue region is given by $x \geq 1$. Here we have $t = s, x = \xi$ and hence $u(x,t) = g(\xi) = 0$. Combining the above, the solution in the time interval $0 \leq t \leq 1/3$ is given by

$$u(x,t) = \begin{cases} 0 & |x| > 1, 0 \leq t \leq 1/3, \\ 3(1+x)/(1+3t) & -1 < x \leq 3t, 0 \leq t \leq 1/3, \\ 3(1-x)/(1-3t) & 3t < x < 1, 0 \leq t \leq 1/3. \end{cases}$$

The graphs of $u(x,t)$ vs x at times $t = 0, 1/9, 2/9, 1/3$ are given below:



Part (b): *Find the law of motion for the shock.*
From the solution obtained in part (a), we see that a shock is developed at $(x,t) = (1, 1/3)$. Suppose that the curve $x = a(t)$ gives the shock trajectory for $t \geq 1/3$, which is the black curve in the figure below:



By the Rankine-Hugoniot condition, the shock speed is determined by

$$a'(t) = \frac{\frac{1}{2}u_+(t)^2 - \frac{1}{2}u_-(t)^2}{u_+(t) - u_-(t)} = \frac{1}{2}(u_+(t) + u_-(t)),$$

where $u_+(t)$ and $u_-(t)$ are the one-sided limits of $u(x,t)$ at the shock.

$$u_+(t) = \lim_{x \to a(t)+} u(x,t), \quad u_-(t) = \lim_{x \to a(t)-} u(x,t).$$

In the present problem, $u_+(t)$ is determined by blue characteristics:

$$u_+(t) = 0,$$

while $u_-(t)$ is determined by red characteristics:

$$u_-(t) = \lim_{x \to a(t)-} \frac{3(1+x)}{1+3t} = \frac{3(1+a(t))}{1+3t}.$$

Thus,

$$a'(t) = \frac{1}{2}\left(0 + \frac{3(1+a(t))}{1+3t}\right) = \frac{3(1+a(t))}{2(1+3t)}.$$

This differential equation gives the law of motion for the shock.

Part (c): *Find the shock trajectory and subsequently the solution $u(x,t)$ for $t > 1/3$.*
Once the shock curve $x = a(t)$ is solved, the solution u is easy to obtain: from the computation in (a) and the picture of global characteristics, we immediately see that

- In the grey region $\{x \leq -1, t > 1/3\}$ and the blue region $\{x > a(t), t > 1/3\}$, solution $u \equiv 0$.
- In the red region $\{-1 < x < a(t), t > 1/3\}$, solution is given by

$$u(x,t) = \frac{3(1+x)}{1+3t}.$$

This way, the solution $u(x,t)$ for $t > 1/3$ will be determined.

It remains to solve $a(t)$ from the following initial value problem of ODE:

$$\frac{da}{dt} = \frac{3(1+a(t))}{2(1+3t)}, \quad a(1/3) = 1.$$

This ODE can be solved by two methods: The method of separation of variables and the method of integrating factor both work. We solve by separation of variables below:

$$\frac{da}{1+a} = \frac{3dt}{2(1+3t)}.$$

Integrate:

$$\ln(1+a) = \frac{1}{2}\ln(1+3t) + C.$$

Use the initial condition $a(1/3) = 1$:

$$\ln(1+1) = \frac{1}{2}\ln\left(1 + 3\cdot\frac{1}{3}\right) + C \Rightarrow C = \frac{1}{2}\ln 2.$$

Hence,

$$\ln(1+a) = \frac{1}{2}\ln(1+3t) + \frac{1}{2}\ln 2 = \frac{1}{2}\ln\left\{2(1+3t)\right\} \Rightarrow 1 + a = \sqrt{2(1+3t)}$$

Thus, the shock curve is

$$\Rightarrow a(t) = \sqrt{2(1+3t)} - 1,$$

and the solution $u(x,t)$ for $t > 1/3$ is given by

$$u(x,t) = \begin{cases} 0 & x \le -1,\ t > 1/3, \\ 3(x+1)/(1+3t) & -1 < x < \sqrt{2(1+3t)} - 1,\ t > 1/3, \\ 0 & x > \sqrt{2(1+3t)} - 1,\ t > 1/3. \end{cases}$$

Combined with Part (a), the solution $u(x,t)$ is now completely solved for all times $t \ge 0$.

Exercises

[1] Consider

$$u_t + (1-u)u_x = 0, \qquad x \in \mathbb{R}, t > 0,$$

$$u(x,0) = g(x) = \begin{cases} 3 & x \leq 0, \\ 3+2x & 0 \leq x \leq 1, \\ 5 & x \geq 1. \end{cases}$$

(a) Find the solution formula for small time $t > 0$. Sktech the characteristics emanating from points on the x-axis. Graph $u(x,t)$ vs x for a few specfic values of time t, before it becomes discontinuous.

(b) Find the smallest time when two distinct characteristics intersect. In other words, find the smallest time when the solution $u(x,t)$ develops a shock (i.e., becomes discontinuous). Graph $u(x,t)$ vs x at that particular moment t.

(c) What is the shock speed in this solution, after the shock is developed? Skecth the shock trajectory on the (x,t) plane. Sktech the characteristics beyond the shock formation.

(d) Find the solution formula for all $x \in \mathbb{R}$ and $t > 0$. Graph $u(x,t)$ vs x, for a few specfic values of time t, after the shock formation.

[2] Consider

$$u_t + (1-u)u_x = 0, \qquad x \in \mathbb{R}, t > 0,$$

$$u(x,0) = g(x) = \begin{cases} 3 & x \leq 0, \\ 3+2x & 0 \leq x \leq 1, \\ 10-5x & 1 \leq x \leq 2, \\ -1+2/x & x \geq 2. \end{cases}$$

Find the time of shock formation. Find the shock trajectory on the (x,t)-plane. Find the solution $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$.

ANSWERS

[1] A shock is developed at $(x,t)=(-1,1/2)$. The shock trajectory is

$$x = a(t) = 1/2 - 3t \qquad (t \geq 1/2).$$

The global solution is: before the shock formation, $0 \leq t < 1/2$,

$$u(x,t) = \begin{cases} 3 & \text{for } x \leq -2t, 0 \leq t < 1/2, \\ (3+2x-2t)/(1-2t) & \text{for } -2t < x < 1-4t, 0 \leq t < 1/2, \\ 5 & \text{for } x \geq 1-4t, 0 \leq t < 1/2; \end{cases}$$

and after the shock formation, $t \geq 1/2$,

$$u(x,t) = \begin{cases} 3 & \text{for } x < 1/2-3t, t \geq 1/2, \\ 5 & \text{for } x > 1/2-3t, t \geq 1/2. \end{cases}$$

I'll omit the graphs.

[2] A shock is developed at $(x,t)=(-1,1/2)$. The shock curve is given by

$$x = a(t) = \frac{7}{5} - 2t - \frac{1}{5}\sqrt{14(1+5t)} \qquad (t \geq 1/2).$$

The global solution is: before the shock formation, $0 \leq t < 1/2$,

$$u(x,t) = \begin{cases} 3 & \text{for } x \leq -2t, 0 \leq t < 1/2, \\ (3+2x-2t)/(1-2t) & \text{for } -2t < x \leq 1-4t, 0 \leq t < 1/2, \\ 5(2-x+t)/(1+5t) & \text{for } 1-4t < x \leq 2+t, 0 \leq t < 1/2, \\ -1 + \dfrac{4}{x-2t+\sqrt{(x-2t)^2+8t}} & \text{for } x > 2+t, 0 \leq t < 1/2; \end{cases}$$

and after the shock formation, $t \geq 1/2$,

$$u(x,t) = \begin{cases} 3 & \text{for } x < a(t), t \geq 1/2, \\ 5(2-x+t)/(1+5t) & \text{for } a(t) < x \leq 2+t, t \geq 1/2, \\ -1 + \dfrac{4}{x-2t+\sqrt{(x-2t)^2+8t}} & \text{for } x > 2+t, t \geq 1/2. \end{cases}$$