# Nonhomogeneous Linear Systems of Differential Equations: the method of variation of parameters

Xu-Yan Chen

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.
- **Solution structure:** The general solutions of the nonhomog system $(*)_{nh}$ are of the form:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_c(t),$$

where $\vec{\mathbf{x}}_p(t)$ is a particular solution of of nonhomog system $(*)_{nh}$, and $\vec{\mathbf{x}}_c(t)$ are general solutions of the homog system

$$(*)_h \qquad \frac{d\vec{\mathbf{x}}_c}{dt} = A(t)\vec{\mathbf{x}}_c.$$

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.
- **Solution structure:** The general solutions of the nonhomog system $(*)_{nh}$ are of the form:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_c(t),$$

where $\vec{\mathbf{x}}_p(t)$ is a particular solution of of nonhomog system $(*)_{nh}$, and $\vec{\mathbf{x}}_c(t)$ are general solutions of the homog system

$$(*)_{h} \qquad \frac{d\vec{\mathbf{x}}_c}{dt} = A(t)\vec{\mathbf{x}}_c.$$

---

- **What is this note about?**

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.
- **Solution structure:** The general solutions of the nonhomog system $(*)_{nh}$ are of the form:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_c(t),$$

where $\vec{\mathbf{x}}_p(t)$ is a particular solution of of nonhomog system $(*)_{nh}$, and $\vec{\mathbf{x}}_c(t)$ are general solutions of the homog system

$$(*)_h \qquad \frac{d\vec{\mathbf{x}}_c}{dt} = A(t)\vec{\mathbf{x}}_c.$$

---

- **What is this note about?**

**Variation of Parameters.**

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.
- **Solution structure:** The general solutions of the nonhomog system $(*)_{nh}$ are of the form:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_c(t),$$

where $\vec{\mathbf{x}}_p(t)$ is a particular solution of of nonhomog system $(*)_{nh}$, and $\vec{\mathbf{x}}_c(t)$ are general solutions of the homog system

$$(*)_h \qquad \frac{d\vec{\mathbf{x}}_c}{dt} = A(t)\vec{\mathbf{x}}_c.$$

---

- **What is this note about?**

**Variation of Parameters.**

If the solutions $\vec{\mathbf{x}}_c(t)$ of the homogeneous system $(*)_h$ have been provided or prepared,

- **Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$$

- **No general method** of solving this class of equations.
- **Solution structure:** The general solutions of the nonhomog system $(*)_{nh}$ are of the form:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_c(t),$$

where $\vec{\mathbf{x}}_p(t)$ is a particular solution of of nonhomog system $(*)_{nh}$, and $\vec{\mathbf{x}}_c(t)$ are general solutions of the homog system

$$(*)_h \qquad \frac{d\vec{\mathbf{x}}_c}{dt} = A(t)\vec{\mathbf{x}}_c.$$

---

- **What is this note about?**

**Variation of Parameters.**

If the solutions $\vec{\mathbf{x}}_c(t)$ of the homogeneous system $(*)_h$ have been provided or prepared, $\Rightarrow$ we can solve the nonhomog system $(*)_{nh}$.

# Fundamental Matrices of Homogeneous Linear Systems:

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

# Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.
- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.

# Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:
$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$
where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.
- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.
- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:
$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.
- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.
- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

  Since

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = M(t)\vec{\mathbf{C}}$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

$$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix.*

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

$$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

  Since

$$\begin{aligned} \vec{\mathbf{x}}_c(t_0) = M(t_0)\vec{\mathbf{C}} = \vec{\mathbf{a}} \quad &\Rightarrow \quad \vec{\mathbf{C}} = M(t_0)^{-1}\,\vec{\mathbf{a}} \\ &\Rightarrow \quad \vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}. \end{aligned}$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.
- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.
- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

$$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

$$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

- $M'(t) = A(t)M(t).$

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

$$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix.*

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

$$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

$$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

- $M'(t) = A(t)M(t).$

- $M(t)$ is an invertible matrix for any $t$.

## Fundamental Matrices of Homogeneous Linear Systems:

- Suppose that the solutions of the homog sys $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ are:

  $$\vec{\mathbf{x}}_c(t) = C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t),$$

  where $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are two linearly independent solutions.

- The matrix $M(t) = \begin{bmatrix} \vec{\mathbf{x}}_1(t) & \vec{\mathbf{x}}_2(t) \end{bmatrix}$ is called a *fundamental matrix*.

- The general solutions of $\vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c$ can be expressed as:

  $$\vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}.$$

- The solution of the I.V.P. $\quad \vec{\mathbf{x}}_c{}' = A(t)\vec{\mathbf{x}}_c, \quad \vec{\mathbf{x}}_c(t_0) = \vec{\mathbf{a}}$:

  $$\vec{\mathbf{x}}_c(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{a}}.$$

- $M'(t) = A(t)M(t)$.

- $M(t)$ is an invertible matrix for any $t$.

- The fundamental matrices are not unique.

**Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

**The Method of Variation of Parameters:**

**Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

## The Method of Variation of Parameters:

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

**Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

**The Method of Variation of Parameters:**

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

- Set $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

## Nonhomogeneous Linear Systems of Differential Equations:

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

## The Method of Variation of Parameters:

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

- Set $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

- Substitute this in $(*)_{nh}$. It simplifies to $\quad \vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

## Nonhomogeneous Linear Systems of Differential Equations:

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

## The Method of Variation of Parameters:

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

- Set $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

- Substitute this in $(*)_{nh}$. It simplifies to $\quad \vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

  *Proof*: $\big(M(t)\vec{\mathbf{u}}(t)\big)' = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t)$
  $\Rightarrow M'(t)\vec{\mathbf{u}}(t) + M(t)\vec{\mathbf{u}}\,'(t) = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t).$
  Since $M'(t) = A(t)M(t)$, we obtain $M(t)\vec{\mathbf{u}}\,'(t) = \vec{\mathbf{f}}(t).$
  Take the inverse: $\vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

**Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

## The Method of Variation of Parameters:

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

- Set $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

- Substitute this in $(*)_{nh}$. It simplifies to $\quad \vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

  *Proof*: $\quad \big(M(t)\vec{\mathbf{u}}(t)\big)' = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t)$

  $\Rightarrow M'(t)\vec{\mathbf{u}}(t) + M(t)\vec{\mathbf{u}}\,'(t) = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t).$

  Since $M'(t) = A(t)M(t)$, we obtain $M(t)\vec{\mathbf{u}}\,'(t) = \vec{\mathbf{f}}(t).$

  Take the inverse: $\vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

- Integrate to get $\vec{\mathbf{u}}(t).$

**Nonhomogeneous Linear Systems of Differential Equations:**

$$(*)_{nh} \qquad \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t).$$

## The Method of Variation of Parameters:

- Suppose that the homogeneous system $\vec{\mathbf{x}}_c' = A(t)\vec{\mathbf{x}}_c$ is solved,

  with $\begin{cases} \text{a fundamental matrix } M(t), \\ \text{the complementary solutions } \vec{\mathbf{x}}_c(t) = M(t)\vec{\mathbf{C}}. \end{cases}$

- Set $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

- Substitute this in $(*)_{nh}$. It simplifies to $\quad \vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

  *Proof*: $\quad \big(M(t)\vec{\mathbf{u}}(t)\big)' = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t)$

  $\Rightarrow M'(t)\vec{\mathbf{u}}(t) + M(t)\vec{\mathbf{u}}\,'(t) = A(t)M(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{f}}(t).$

  Since $M'(t) = A(t)M(t)$, we obtain $M(t)\vec{\mathbf{u}}\,'(t) = \vec{\mathbf{f}}(t).$

  Take the inverse: $\vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t).$

- Integrate to get $\vec{\mathbf{u}}(t)$.

- Finally, multiply to get $\quad \vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t).$

**Example 1:** Solve $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

---

**Example 1:** Solve $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

---

- Eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (details skipped here)

  $\Rightarrow$ Complementary solutions: $\vec{\mathbf{x}}_c(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\Rightarrow$ A fundamental matrix: $M(t) = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}$

## Example 1: Solve $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

- Eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (details skipped here)

  $\Rightarrow$ Complementary solutions: $\vec{\mathbf{x}}_c(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\Rightarrow$ A fundamental matrix: $M(t) = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}$

- Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$

**Example 1:** Solve $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t} \tan t \\ 2 + 30t + 2e^{3t} \tan t \end{bmatrix}$

---

- Eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (details skipped here)

  $\Rightarrow$ Complementary solutions: $\vec{\mathbf{x}}_c(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\Rightarrow$ A fundamental matrix: $M(t) = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}$

- Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$
- The original system is simplified to $\vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t)$:

$$
\begin{aligned}
\vec{\mathbf{u}}\,'(t) &= \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}^{-1} \begin{bmatrix} -1 + 21t + e^{3t} \tan t \\ 2 + 30t + 2e^{3t} \tan t \end{bmatrix} \\
&= \frac{1}{-4e^{2t}} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ -2e^{-t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 + 21t + e^{3t} \tan t \\ 2 + 30t + 2e^{3t} \tan t \end{bmatrix} \\
&= \begin{bmatrix} (1 - 3t)e^{t} \\ 18te^{-3t} + \tan t \end{bmatrix}.
\end{aligned}
$$

## Example 1 (continued): $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t} \tan t \\ 2 + 30t + 2e^{3t} \tan t \end{bmatrix}$

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} (1-3t)e^t \\ 18te^{-3t} + \tan t \end{bmatrix}$:

$$\begin{aligned} &\int (1-3t)e^t dt = (1-3t)e^t - \int (-3)e^t dt = (1-3t)e^t + 3e^t + C, \\ &\int 18te^{-3t} dt = -6te^{-3t} - \int (-6)e^{-3t} dt = -6te^{-3t} - 2e^{-3t} + C, \\ &\int \tan t \; dt = -\ln|\cos t| + C. \end{aligned}$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} (4-3t)e^t + C_1 \\ (-2-6t)e^{-3t} - \ln|\cos t| + C_2 \end{bmatrix}.$$

## Example 1 (continued): $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} (1-3t)e^{t} \\ 18te^{-3t} + \tan t \end{bmatrix}$:

$$\begin{aligned}
&\textstyle\int (1-3t)e^{t}dt = (1-3t)e^{t} - \int(-3)e^{t}dt = (1-3t)e^{t} + 3e^{t} + C, \\
&\textstyle\int 18te^{-3t}dt = -6te^{-3t} - \int(-6)e^{-3t}dt = -6te^{-3t} - 2e^{-3t} + C, \\
&\textstyle\int \tan t\ dt = -\ln|\cos t| + C.
\end{aligned}$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} (4-3t)e^{t} + C_1 \\ (-2-6t)e^{-3t} - \ln|\cos t| + C_2 \end{bmatrix}.$$

- Finally, the solutions $\vec{\mathbf{x}}(t)$ are obtained from $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$:

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} (4-3t)e^{t} + C_1 \\ (-2-6t)e^{-3t} - \ln|\cos t| + C_2 \end{bmatrix} \\
&= \begin{bmatrix} -6 - 3t - e^{3t}\ln|\cos t| \\ \\ 4 - 18t - 2e^{3t}\ln|\cos t| \end{bmatrix} + C_1 e^{-t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

**Example 2:** Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

**Example 2:** Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

---

- Set $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$2y'' - y' - y = \tfrac{3e^{\frac{3}{2}t}}{1+e^t} \Leftrightarrow \vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix}$$

**Example 2:** Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

- Set $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

  $$2y'' - y' - y = \frac{3e^{\frac{3}{2}t}}{1+e^t} \Leftrightarrow \vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix}$$

- Characteristic roots $\Rightarrow y_c(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^t$

**Example 2:** Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

- Set $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$2y'' - y' - y = \tfrac{3e^{\frac{3}{2}t}}{1+e^t} \Leftrightarrow \vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix}$$

- Characteristic roots $\Rightarrow y_c(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^t$
- $y_c(t) \Rightarrow$ complem. solutions of the system & a fund. matrix:

$$\vec{\mathbf{x}}_c(t) = C_1 \begin{bmatrix} e^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} \end{bmatrix} + C_2 \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \qquad M(t) = \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix}.$$

**Example 2:** Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

- Set $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$2y'' - y' - y = \tfrac{3e^{\frac{3}{2}t}}{1+e^t} \Leftrightarrow \vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix}$$

- Characteristic roots $\Rightarrow y_c(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^t$
- $y_c(t) \Rightarrow$ complem. solutions of the system & a fund. matrix:

$$\vec{\mathbf{x}}_c(t) = C_1 \begin{bmatrix} e^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} \end{bmatrix} + C_2 \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \qquad M(t) = \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix}.$$

- Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$. The system simplifies to:

$$\begin{aligned} \vec{\mathbf{u}}\,'(t) &= M(t)^{-1}\vec{\mathbf{f}}(t) = \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix} \\ &= \frac{1}{\frac{3}{2}e^{\frac{1}{2}t}} \begin{bmatrix} e^t & -e^t \\ \frac{1}{2}e^{-\frac{1}{2}t} & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix} = \begin{bmatrix} -\frac{e^{2t}}{1+e^t} \\ \frac{e^{\frac{1}{2}t}}{1+e^t} \end{bmatrix}. \end{aligned}$$

## Example 2 (continued): $2y'' - y' - y = \frac{3e^{\frac{3}{2}t}}{1+e^t}$

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} -\frac{e^{2t}}{1+e^t} \\ \frac{e^{\frac{1}{2}t}}{1+e^t} \end{bmatrix}$:

$$\int \frac{e^{2t}}{1+e^t} dt = \int \frac{s}{1+s} ds = \int (1 - \frac{1}{1+s}) ds = s - \ln|1+s| + C$$

$$\text{(substituted } s = e^t\text{)},$$

$$\int \frac{e^{\frac{1}{2}t}}{1+e^t} dt = \int \frac{2}{1+s^2} ds = 2\arctan s + C \qquad \text{(substituted } s = e^{t/2}\text{)}.$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} -e^t + \ln(1+e^t) + C_1 \\ 2\arctan(e^{\frac{1}{2}t}) + C_2 \end{bmatrix}.$$

## Example 2 (continued): $2y'' - y' - y = \frac{3e^{\frac{3}{2}t}}{1+e^t}$

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} -\frac{e^{2t}}{1+e^t} \\ \frac{e^{\frac{1}{2}t}}{1+e^t} \end{bmatrix}$:

$$\int \frac{e^{2t}}{1+e^t} dt = \int \frac{s}{1+s} ds = \int (1 - \frac{1}{1+s}) ds = s - \ln|1+s| + C$$
$$\text{(substituted } s = e^t\text{)},$$
$$\int \frac{e^{\frac{1}{2}t}}{1+e^t} dt = \int \frac{2}{1+s^2} ds = 2\arctan s + C \qquad \text{(substituted } s = e^{t/2}\text{)}.$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} -e^t + \ln(1+e^t) + C_1 \\ 2\arctan(e^{\frac{1}{2}t}) + C_2 \end{bmatrix}.$$

- Finally, the solutions $\vec{\mathbf{x}}(t)$ are obtained from $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$:

$$\begin{aligned} \vec{\mathbf{x}}(t) &= \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix} \begin{bmatrix} -e^t + \ln(1+e^t) + C_1 \\ 2\arctan(e^{\frac{1}{2}t}) + C_2 \end{bmatrix} = \cdots, \\ y(t) &= x_1(t) \\ &= -e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\ln(1+e^t) + 2e^t\arctan(e^{\frac{1}{2}t}) + C_1 e^{-\frac{1}{2}t} + C_2 e^t. \end{aligned}$$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $\quad (1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $\qquad y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h \;\; (1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0.$

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h$ $(1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

$$(1+t)(e^t z)'' + (-1-2t)(e^t z)' + te^t z = 0,$$
$$(1+t)(e^t z'' + 2e^t z' + e^t z) + (-1-2t)(e^t z' + e^t z) + te^t z = 0,$$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h$ $(1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

$$(1+t)(e^t z)'' + (-1-2t)(e^t z)' + te^t z = 0,$$
$$(1+t)(e^t z'' + 2e^t z' + e^t z) + (-1-2t)(e^t z' + e^t z) + te^t z = 0,$$
$$(1+t)e^t z'' + e^t z' = 0,$$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $\quad (1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0.$

---

- Reduction of Order: $\quad y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h \;\; (1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$

  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

$$(1+t)(e^t z)'' + (-1-2t)(e^t z)' + te^t z = 0,$$
$$(1+t)(e^t z'' + 2e^t z' + e^t z) + (-1-2t)(e^t z' + e^t z) + te^t z = 0,$$
$$(1+t)e^t z'' + e^t z' = 0,$$
$$(*)_z \qquad (1+t)z'' + z' = 0.$$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h \quad (1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

    $(*)_z \qquad (1+t)z'' + z' = 0.$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h$ $(1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

    $(*)_z \qquad (1+t)z'' + z' = 0.$
  - Set $w(t) = z'(t)$. The eq $(*)_z$ becomes a 1st order linear eq:

    $(*)_w \qquad (1+t)w' + w = 0.$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $\quad (1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0.$

---

- Reduction of Order: $\quad y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h \; (1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

    $(*)_z \qquad (1+t)z'' + z' = 0.$
  - Set $w(t) = z'(t)$. The eq $(*)_z$ becomes a 1st order linear eq:

    $(*)_w \qquad (1+t)w' + w = 0.$
  - Solve $(*)_w \Rightarrow w(t) = \dfrac{C_1}{1+t}$.

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $\quad (1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $\qquad y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h \;\; (1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

    $$(*)_z \qquad (1+t)z'' + z' = 0.$$

  - Set $w(t) = z'(t)$. The eq $(*)_z$ becomes a 1st order linear eq:

    $$(*)_w \qquad (1+t)w' + w = 0.$$

  - Solve $(*)_w \Rightarrow w(t) = \dfrac{C_1}{1+t}$.
  - $z' = w \quad \Rightarrow \; z(t) = \int w(t)dt = \int \dfrac{C_1}{1+t}dt = C_1 \ln|1+t| + C_2$

**Example 3:** Solve $(1+t)y'' + (-1-2t)y' + ty = \frac{e^t}{1+t}$,

given that $y_0(t) = e^t$ satisfies $(1+t)y_0'' + (-1-2t)y_0' + ty_0 = 0$.

---

- Reduction of Order: $y_0(t) \Rightarrow y_c(t)$.
- Variation of Parameters: $y_c(t) \Rightarrow$ Solutions of the nonhomog eq.

---

- **Reduction of Order: Solve** $(*)_h$ $(1+t)y_c'' + (-1-2t)y_c' + ty_c = 0$
  - Set $y_c(t) = y_0(t)z(t) = e^t z(t)$ and substitute $y(t) = e^t z(t)$ in $(*)_h$:

    $(*)_z \qquad (1+t)z'' + z' = 0.$
  - Set $w(t) = z'(t)$. The eq $(*)_z$ becomes a 1st order linear eq:

    $(*)_w \qquad (1+t)w' + w = 0.$
  - Solve $(*)_w \Rightarrow w(t) = \frac{C_1}{1+t}$.
  - $z' = w \quad \Rightarrow \quad z(t) = \int w(t)dt = \int \frac{C_1}{1+t}dt = C_1 \ln|1+t| + C_2$
  - $y_c(t) = y_0(t)z(t) = e^t z(t) = e^t\left[C_1 \ln|1+t| + C_2\right]$.

    Or, equivalently, $y_c(t) = C_1 e^t \ln|1+t| + C_2 e^t$.

**Example 3 (continued):** $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}.$

---

- Reduction of Order: $y_0(t) = e^t \Rightarrow y_c(t) = C_1 e^t \ln|1+t| + C_2 e^t.$

---

- **Variation of Parameters:** $y_c(t) \Rightarrow$ **Solve the nonhomog eq.**

**Example 3 (continued):** $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$.

---

- Reduction of Order: $y_0(t) = e^t \Rightarrow y_c(t) = C_1 e^t \ln|1+t| + C_2 e^t$.

---

- **Variation of Parameters:** $y_c(t) \Rightarrow$ **Solve the nonhomog eq.**
  - $y_c(t)$ gives a fundamental matrix:

$$M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^t \ln|1+t| & e^t \\ e^t \ln|1+t| + \frac{e^t}{1+t} & e^t \end{bmatrix}$$

## Example 3 (continued): $(1+t)y'' + (-1-2t)y' + ty = \frac{e^t}{1+t}.$

- Reduction of Order: $y_0(t) = e^t \Rightarrow y_c(t) = C_1 e^t \ln|1+t| + C_2 e^t.$

- **Variation of Parameters:** $y_c(t) \Rightarrow$ **Solve the nonhomog eq.**
  - $y_c(t)$ gives a fundamental matrix:
    $$M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^t \ln|1+t| & e^t \\ e^t \ln|1+t| + \frac{e^t}{1+t} & e^t \end{bmatrix}$$
  - $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$ satisfies $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix}.$

**Example 3 (continued):** $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}$.

- Reduction of Order: $y_0(t) = e^t \Rightarrow y_c(t) = C_1 e^t \ln|1+t| + C_2 e^t$.

- **Variation of Parameters:** $y_c(t) \Rightarrow$ **Solve the nonhomog eq.**
  - $y_c(t)$ gives a fundamental matrix:

$$M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^t \ln|1+t| & e^t \\ e^t \ln|1+t| + \frac{e^t}{1+t} & e^t \end{bmatrix}$$

  - $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$ satisfies $\vec{\mathbf{x}}\,' = A(t)\vec{\mathbf{x}} + \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix}$.
  - Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$. Then $\vec{\mathbf{u}}(t)$ satisfies $\vec{\mathbf{u}}\,'(t) = M(t)^{-1}\,\vec{\mathbf{f}}(t)$.

$$\begin{aligned} \vec{\mathbf{u}}\,'(t) &= \begin{bmatrix} e^t \ln|1+t| & e^t \\ e^t \ln|1+t| + \frac{e^t}{1+t} & e^t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix} \\ &= \frac{1}{-\frac{e^{2t}}{1+t}} \begin{bmatrix} e^t & -e^t \\ -e^t \ln|1+t| - \frac{e^t}{1+t} & e^t \ln|1+t| \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{1+t} \\ -\frac{\ln|1+t|}{1+t} \end{bmatrix}. \end{aligned}$$

**Example 3 (continued):** $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}.$

## Variation of Parameters (continued)

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} \frac{1}{1+t} \\ -\frac{\ln|1+t|}{1+t} \end{bmatrix}$:

$$\int \tfrac{1}{1+t}dt = \ln|1+t| + C,$$
$$\int \tfrac{\ln|1+t|}{1+t}dt = \int s\,ds = \tfrac{1}{2}s^2 + C \qquad \text{(substituted } s = \ln|1+t|\text{).}$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} \ln|1+t| + C_1 \\ -\frac{1}{2}\left(\ln|1+t|\right)^2 + C_2 \end{bmatrix}.$$

## Example 3 (continued): $(1+t)y'' + (-1-2t)y' + ty = \dfrac{e^t}{1+t}.$

---

### Variation of Parameters (continued)

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} \frac{1}{1+t} \\ -\frac{\ln|1+t|}{1+t} \end{bmatrix}$:

$$\int \tfrac{1}{1+t} dt = \ln|1+t| + C,$$
$$\int \tfrac{\ln|1+t|}{1+t} dt = \int s ds = \tfrac{1}{2}s^2 + C \qquad \text{(substituted } s = \ln|1+t|\text{)}.$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} \ln|1+t| + C_1 \\ -\frac{1}{2}(\ln|1+t|)^2 + C_2 \end{bmatrix}.$$

- Finally, we can get $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$ and $y(t) = x_1(t)$:

$$\begin{aligned} y(t) &= u_1(t)e^t \ln|1+t| + u_2(t)e^t \\ &= \ln|1+t| \cdot e^t \ln|1+t| - \tfrac{1}{2}(\ln|1+t|)^2 \cdot e^t \\ &\quad + C_1 e^t \ln|1+t| + C_2 e^t \\ &= \tfrac{1}{2}e^t (\ln|1+t|)^2 + C_1 e^t \ln|1+t| + C_2 e^t. \end{aligned}$$