Variation of Parameters

Nonhomog. Lin. System in n-dim. (*) $\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t)$.

Step 1 Prepare complementary solutions & a fundamental matrix $M(t)$.
i.e. Solve the homog. system $\vec{x}_c' = A(t)\vec{x}_c$

Step 2. Variation of Parameters. Set $\vec{x}(t) = M(t)\vec{u}(t)$.
The equation (*) $\Leftrightarrow$ $\frac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t)$ (**)

- Integrate to get $\vec{u}_p(t)$: a particular sol of (**).
- Then $\vec{x}_p(t) = M(t)\vec{u}_p(t)$ is a particular sol of (*)

Finally, the general solutions of (*) are $\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t)$.

Example 1 Solve $\vec{x}\,' = A\vec{x} + \vec{f}(t)$, where $A = \begin{bmatrix} -9 & -8 & 3 \\ 8 & 7 & -3 \\ -6 & -6 & 2 \end{bmatrix}$, $\vec{f}(t) = \begin{bmatrix} \frac{1-3t}{1+t}e^{-t} + (2-2t)e^{t} \\ \frac{3t}{1+t}e^{-t} + (-1+2t)e^{t} \\ \frac{2}{1+t}e^{-t} + 3e^{t} \end{bmatrix}$

Solution $\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t)$.

Step 1 Find $\vec{x}_c(t)$: Complementary Sol's i.e. Sol's of $\vec{x}_c\,' = A\vec{x}_c$.

- Eigenvalues of A : $\det(A-\lambda I) = \cdots = -\lambda^3 + 3\lambda + 2 = (\lambda-2)(\lambda+1)^2 \Rightarrow \lambda_1 = 2,\ \lambda_2 = \lambda_3 = -1$.

- Eigenvectors for $\lambda_1 = 2$:

$$(A-2I)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -11 & -8 & 3 \\ 8 & 5 & -3 \\ -6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

- Generalized Eigenvectors for $\lambda_2 = \lambda_3 = -1$

$$(A+I)^2\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -18 & -18 & 9 \\ 18 & 18 & -9 \\ -18 & -18 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 + x_2 - \frac{1}{2}x_3 = 0$$

$$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

- Complementary Sol's.

$$\vec{x}_c(t) = c_1 e^{tA}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{tA}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{tA}\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\{I + t(A+I)\}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t}\{I + t(A+I)\}\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$= c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -8 & -8 & 3 \\ 8 & 8 & -3 \\ -6 & -6 & 3 \end{bmatrix}\right\}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -8 & -8 & 3 \\ 8 & 8 & -3 \\ -6 & -6 & 3 \end{bmatrix}\right\}\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$= c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} \frac{1}{2} - t \\ t \\ 1 \end{bmatrix}.$$

Step 2. Variation of Parameters

· A fundamental matrix $M(t) = \begin{bmatrix} e^{2t} & -e^{-t} & (\frac{1}{2}-t)e^{-t} \\ -e^{2t} & e^{-t} & te^{-t} \\ e^{2t} & 0 & e^{-t} \end{bmatrix}$.

· Set $\vec{x}(t) = M(t)\vec{u}(t)$. The eq for $\vec{u}(t)$ is:

$$\vec{u}\,'(t) = M(t)^{-1}\vec{f}(t) = \begin{bmatrix} -2e^{-2t} & -2e^{-2t} & e^{-2t} \\ (-2-2t)e^{t} & (-1-2t)e^{t} & e^{t} \\ 2e^{t} & 2e^{t} & 0 \end{bmatrix} \begin{bmatrix} \frac{1-3t}{1+t}e^{-t} + (2-2t)e^{t} \\ \frac{3t}{1+t}e^{-t} + (-1+2t)e^{t} \\ \frac{2}{1+t}e^{-t} + 3e^{t} \end{bmatrix}$$

$$\vec{u}\,'(t) = \begin{bmatrix} e^{-t} \\ \frac{t}{1+t} \\ \frac{2}{1+t} + 2e^{2t} \end{bmatrix} \xRightarrow[\text{integrate}]{} \vec{u}_p(t) = \begin{bmatrix} -e^{-t} \\ t - \ln|1+t| \\ 2\ln|1+t| + e^{2t} \end{bmatrix}$$

· $\vec{x}_p(t) = M(t)\vec{u}_p(t) = \begin{bmatrix} e^{2t} & -e^{-t} & (\frac{1}{2}-t)e^{-t} \\ -e^{2t} & e^{-t} & te^{-t} \\ e^{2t} & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -e^{-t} \\ t - \ln|1+t| \\ 2\ln|1+t| + e^{2t} \end{bmatrix}$

$$= \begin{bmatrix} e^{t}(-\frac{1}{2}-t) + e^{-t}(-t + 2\ln|1+t| - 2t\ln|1+t|) \\ e^{t}(1+t) + e^{-t}(t - \ln|1+t| + 2t\ln|1+t|) \\ 2e^{-t}\ln|1+t| \end{bmatrix}.$$

Gen. Sol's of $\vec{x}\,' = A\vec{x} + \vec{f}(t)$

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t) = \cdots\cdots + \cdots\cdots$$

Example 2 Solve $4y''' + 9y' = \sec\frac{3t}{2}$ (*)

Solution $y(t) = y_p(t) + y_c(t)$

Step 1 Find $y_c(t)$: $4y_c''' + 9y_c' = 0$

- Eigenvalues: $4\lambda^3 + 9\lambda = \lambda(4\lambda^2+9) \Rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{3}{2}i,\ \lambda_3 = -\frac{3}{2}i$
- $y_c(t) = C_1 + C_2\cos\frac{3t}{2} + C_3\sin\frac{3t}{2}$

Step 2 Convert (*) to a nonhomogeneous linear system:

- Define $\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$. The eq's for x_1, x_2, x_3: $\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = -\frac{9}{4}y' + \frac{1}{4}\sec\frac{3t}{2} = -\frac{9}{4}x_2 + \frac{1}{4}\sec\frac{3t}{2} \end{cases}$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{9}{4} & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4}\sec\frac{3t}{2} \end{bmatrix} \leftarrow \text{Pay attention to the Coeff. } \frac{1}{4}.$$

- Complementary Solutions of the system: $\vec{x}_c(t) = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} \cos\frac{3t}{2} \\ -\frac{3}{2}\sin\frac{3t}{2} \\ -\frac{9}{4}\cos\frac{3t}{2} \end{bmatrix} + C_3\begin{bmatrix} \sin\frac{3t}{2} \\ \frac{3}{2}\cos\frac{3t}{2} \\ -\frac{9}{4}\sin\frac{3t}{2} \end{bmatrix}$

- A fundamental matrix:

$$M(t) = \begin{bmatrix} 1 & \cos\frac{3t}{2} & \sin\frac{3t}{2} \\ 0 & -\frac{3}{2}\sin\frac{3t}{2} & \frac{3}{2}\cos\frac{3t}{2} \\ 0 & -\frac{9}{4}\cos\frac{3t}{2} & -\frac{9}{4}\sin\frac{3t}{2} \end{bmatrix}$$

Step 3. Variation of Parameters

Set $\vec{x}(t) = M(t)\vec{u}(t)$.

The Diff. Eq. for $\vec{u}(t)$ becomes:

$$\vec{u}'(t) = M(t)^{-1}\vec{f}(t) = \begin{bmatrix} 1 & 0 & \frac{4}{9} \\ 0 & -\frac{2}{3}\sin\frac{3t}{2} & -\frac{4}{9}\cos\frac{3t}{2} \\ 0 & \frac{2}{3}\cos\frac{3t}{2} & -\frac{4}{9}\sin\frac{3t}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4}\sec\frac{3t}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{9}\sec\frac{3t}{2} \\ -\frac{1}{9} \\ -\frac{1}{9}\tan\frac{3t}{2} \end{bmatrix}$$

$$\xrightarrow{\text{Integrate}} \quad \vec{u}_p(t) = \begin{bmatrix} \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| \\ -\frac{1}{9}t \\ \frac{2}{27}\ln\left|\cos\frac{3t}{2}\right| \end{bmatrix}$$

$$\Rightarrow \vec{x}_p(t) = M(t)\vec{u}_p(t)$$

$$\Rightarrow y_p(t) = [\text{the first component of } \vec{x}_p(t)]$$

$$= \left(\text{the first component of } \begin{bmatrix} 1 & \cos\frac{3t}{2} & \sin\frac{3t}{2} \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| \\ -\frac{1}{9}t \\ \frac{2}{27}\ln\left|\cos\frac{3t}{2}\right| \end{bmatrix}\right)$$

$$= \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| - \frac{1}{9}t\cos\frac{3t}{2} + \frac{2}{27}\sin\frac{3t}{2}\ln\left|\cos\frac{3t}{2}\right|.$$

Gen. Sol's of $4y''' + 9y' = \sec\frac{3t}{2}$ are:

$$y(t) = y_p(t) + y_c(t) = \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| - \frac{1}{9}t\cos\frac{3t}{2} + \frac{2}{27}\sin\frac{3t}{2}\ln\left|\cos\frac{3t}{2}\right| + C_1 + C_2\cos\frac{3t}{2} + C_3\sin\frac{3t}{2}$$