Variation of Parameters for n-Dim. Nonhomog. Linear Systems

$$(*) \quad \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t)$$

Step 1. Prepare complementary solutions $\vec{x}_c(t)$ & a fundamental matrix $M(t)$.

i.e. Solve the homog. system $\frac{d\vec{x}_c}{dt} = A(t)\vec{x}_c$ to get

$$\vec{x}_c(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + \cdots + C_n\vec{x}_n(t).$$ ⇐ complementary sols.

Let $M(t) = [\vec{x}_1(t) \;\; \vec{x}_2(t) \;\; \cdots \;\; \vec{x}_n(t)]$. (an $n\times n$ invertible matrix)

⇐ a fundamental matrix.

Step 2. Variation of Parameters.

Set $\vec{x}(t) = M(t)\vec{u}(t)$.

The Equation $(*)$ $\Leftrightarrow$ $\frac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t) \quad (**)$

Step 3. Integrate to get $\vec{u}_p(t)$: a particular solution of $(**)$

Step 4. Then $\vec{x}_p(t) = M(t)\vec{u}_p(t)$ is a particular solution of $(*)$,

and the general solutions of $(*)$ are $\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t)$.

Variation of Parameters works, because of the following equivalence:

Set $\vec{x}(t) = M(t)\vec{u}(t)$.

$$(*) \quad \frac{d\vec{x}}{dt} = A(t)\vec{x}(t) + \vec{f}(t) \iff (**) \quad \frac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t)$$

- Derivation of this Equivalence :
  - By definition, $M(t) = [\vec{x}_1(t) \ \ \vec{x}_2(t) \ \cdots \ \vec{x}_n(t)]$,
    where $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are linearly indep. Sol's of the homog. system:
    $$\frac{d\vec{x}_1(t)}{dt} = A(t)\vec{x}_1(t), \quad \dots\dots, \quad \frac{d\vec{x}_n(t)}{dt} = A(t)\vec{x}_n(t).$$
  - $\frac{dM(t)}{dt} = A(t)M(t)$

    Reason: $\frac{dM}{dt}(t) = \left[\frac{d\vec{x}_1}{dt}(t) \ \ \frac{d\vec{x}_2}{dt}(t) \ \cdots \ \frac{d\vec{x}_n}{dt}(t)\right] =$
    $$= [A(t)\vec{x}_1(t) \ \ A(t)\vec{x}_2(t) \ \cdots \ A(t)\vec{x}_n(t)] =$$
    $$= A(t)[\vec{x}_1(t) \ \ \vec{x}_2(t) \ \cdots \ \vec{x}_n(t)] = A(t)M(t)$$
  - Substitute $\vec{x}(t) = M(t)\vec{u}(t)$ in $(*)$:
    $$\frac{d}{dt}\{M(t)\vec{u}(t)\} = A(t)M(t)\vec{u}(t) + \vec{f}(t)$$
    $$= \frac{dM(t)}{dt}\vec{u}(t) + M(t)\frac{d\vec{u}}{dt} = A(t)M(t)\vec{u}(t) + M(t)\frac{d\vec{u}}{dt}$$
    $$\iff M(t)\frac{d\vec{u}}{dt} = \vec{f}(t) \iff (**)$$

Example 1: Solve $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

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- Eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (details skipped here)

  $\Rightarrow$ Complementary solutions: $\vec{\mathbf{x}}_c(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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- Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$

Example 1: Solve $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

- Eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (details skipped here)

  $\Rightarrow$ Complementary solutions: $\vec{\mathbf{x}}_c(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\Rightarrow$ A fundamental matrix: $M(t) = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}$

- Set $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$
- The original system is simplified to $\vec{\mathbf{u}}'(t) = M(t)^{-1}\vec{\mathbf{f}}(t)$:

$$\begin{aligned} \vec{\mathbf{u}}'(t) &= \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}^{-1} \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix} \\ &= \frac{1}{-4e^{2t}} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ -2e^{-t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix} \\ &= \begin{bmatrix} (1-3t)e^{t} \\ 18te^{-3t} + \tan t \end{bmatrix}. \end{aligned}$$

Example 1 (continued): $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t}\tan t \\ 2 + 30t + 2e^{3t}\tan t \end{bmatrix}$

- Integrate $\vec{\mathbf{u}}\,'(t) = \begin{bmatrix} (1-3t)e^{t} \\ 18te^{-3t} + \tan t \end{bmatrix}$:

$$\begin{aligned}
&\int (1-3t)e^{t}\,dt = (1-3t)e^{t} - \int (-3)e^{t}\,dt = (1-3t)e^{t} + 3e^{t} + C, \\
&\int 18te^{-3t}\,dt = -6te^{-3t} - \int (-6)e^{-3t}\,dt = -6te^{-3t} - 2e^{-3t} + C, \\
&\int \tan t\ dt = -\ln|\cos t| + C.
\end{aligned}$$

$$\Rightarrow \vec{\mathbf{u}}(t) = \begin{bmatrix} (4-3t)e^{t} + C_1 \\ (-2-6t)e^{-3t} - \ln|\cos t| + C_2 \end{bmatrix}.$$

Example 1 (continued): $\vec{\mathbf{x}}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} -1 + 21t + e^{3t} \tan t \\ 2 + 30t + 2e^{3t} \tan t \end{bmatrix}$

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- Finally, the solutions $\vec{\mathbf{x}}(t)$ are obtained from $\vec{\mathbf{x}}(t) = M(t)\vec{\mathbf{u}}(t)$:

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} (4-3t)e^t + C_1 \\ (-2-6t)e^{-3t} - \ln|\cos t| + C_2 \end{bmatrix}\\
&= \begin{bmatrix} -6 - 3t - e^{3t}\ln|\cos t| \\ \\ 4 - 18t - 2e^{3t}\ln|\cos t| \end{bmatrix} + C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

Example Solve (*) $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1-\cos(2t) & -1-\sin(2t) \\ 1-\sin(2t) & -1+\cos(2t) \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$,

given the complementary solutions $\vec{x}_c(t) = C_1 e^{-2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + C_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$.

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Solution. The given $\vec{x}_c(t) \Rightarrow$ a Fundamental Matrix $M(t) = [\quad]$

- Set $\vec{x}(t) = M(t)\vec{u}(t)$. Eq (*) becomes:

(**) $\frac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t)$

- Integrate $\Rightarrow \vec{u}_p(t) =$ [a particular sol of (**)]

- $\Rightarrow \vec{x}_p(t) = M(t)\vec{u}_p(t) =$

[a particular sol. of (*)]

- **General Sol's of (*)**:

$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t) =$

Example Solve (*) $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -1-\cos(2t) & -1-\sin(2t) \\ 1-\sin(2t) & -1+\cos(2t) \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$,

given the complementary solutions $\vec{x}_c(t) = C_1 e^{-2t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + C_2\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$.

Solution • The given $\vec{x}_c(t) \Rightarrow$ a Fundamental Matrix $M(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$

• Set $\vec{x}(t) = M(t)\vec{u}(t)$. Eq (*) becomes:

(**) $\dfrac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t) = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} \sin t \\ e^{-2t}\cos t \end{bmatrix}$.

• Integrate $\Rightarrow \vec{u}_p(t) = \begin{bmatrix} -\cos t \\ -\frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t \end{bmatrix}$ [a particular sol of (**)]

• $\Rightarrow \vec{x}_p(t) = M(t)\vec{u}_p(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}\begin{bmatrix} -\cos t \\ -\frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t \end{bmatrix}$

$$= e^{-2t}\begin{bmatrix} -\cos^2 t + \frac{2}{5}\cos t\sin t - \frac{1}{5}\sin^2 t \\ -\frac{2}{5}\cos^2 t - \frac{4}{5}\cos t\sin t \end{bmatrix}$$ [a particular sol. of (*)]

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given the complementary solutions $\vec{x}_c(t) = C_1 e^{-2t}\begin{bmatrix}\cos t \\ \sin t\end{bmatrix} + C_2\begin{bmatrix}-\sin t \\ \cos t\end{bmatrix}$.

Solution. The given $\vec{x}_c(t) \Rightarrow$ a Fundamental Matrix $M(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$

- Set $\vec{x}(t) = M(t)\vec{u}(t)$. Eq (*) becomes:

(**) $$\frac{d\vec{u}}{dt} = M(t)^{-1}\vec{f}(t) = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ e^{-2t}\end{bmatrix} = \begin{bmatrix} \sin t \\ e^{-2t}\cos t \end{bmatrix}.$$

- Integrate $\Rightarrow$ $\vec{u}_p(t) = \begin{bmatrix} -\cos t \\ -\frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t \end{bmatrix}$ [a particular sol of (**)]

- $\Rightarrow \vec{x}_p(t) = M(t)\vec{u}_p(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}\begin{bmatrix} -\cos t \\ -\frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t \end{bmatrix}$

$$= e^{-2t}\begin{bmatrix} -\cos^2 t + \frac{2}{5}\cos t\sin t - \frac{1}{5}\sin^2 t \\ -\frac{2}{5}\cos^2 t - \frac{4}{5}\cos t\sin t \end{bmatrix}$$ [a particular sol. of (*)]

- **General Sol's of (*):**

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t) = e^{-2t}\begin{bmatrix} -\cos^2 t + \frac{2}{5}\cos t\sin t - \frac{1}{5}\sin^2 t \\ -\frac{2}{5}\cos^2 t - \frac{4}{5}\cos t\sin t \end{bmatrix} + C_1 e^{-2t}\begin{bmatrix}\cos t \\ \sin t\end{bmatrix} + C_2\begin{bmatrix}-\sin t \\ \cos t\end{bmatrix}.$$

Variation of Parameters for n-th Order Nonhomog. Lin. Diff Eq's.

(*) $a_n(t)\dfrac{d^n y}{dt^n} + a_{n-1}(t)\dfrac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\dfrac{dy}{dt} + a_0(t)y = f(t)$.

Step 1. Prepare complementary solutions $y_c(t)$ & a fundamental matrix $M(t)$.

i.e. Solve the homog. eq. $a_n(t)\dfrac{d^n y_c}{dt^n} + a_{n-1}(t)\dfrac{d^{n-1}y_c}{dt} + \cdots + a_1(t)\dfrac{dy_c}{dt} + a_0(t)y_c = 0$,

to get $y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$. ⇐ Complementary solutions

Let $M(t) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$ ⇐ the Wronskian of $y_1, y_2, \cdots, y_n$ gives a fund. matrix $M(t)$.

Step 2. Set $\vec{x}(t) = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix} = M(t)\vec{u}(t)$.

Eq (*) $\Longleftrightarrow$ $\dfrac{d\vec{u}}{dt} = M(t)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t)/a_n(t) \end{bmatrix}$ ← Note that it is divided by $a_n(t)$.

Step 3. Integrate to get $\vec{u}(t)$.

Step 4. $y(t) = y_1(t)u_1(t) + y_2(t)u_2(t) + \cdots + y_n(t)u_n(t)$.

Example 2: Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

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- Eigenvalues: $2\lambda^2 - \lambda - 1 = (2\lambda+1)(\lambda-1) \Rightarrow \lambda_1 = -1/2, \lambda_2 = 1$
  $\Rightarrow$ Complementary solutions: $y_c(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^t$ with
  $y_1(t) = e^{-\frac{1}{2}t}, y_2(t) = e^t$.

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- A fundamental matrix (the Wronskian of y_1, y_2):

$$M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix}.$$

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- A fundamental matrix (the Wronskian of y_1, y_2):

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- Set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = M(t)\vec{\mathbf{u}}(t)$. The Diff Eq for $\vec{\mathbf{u}}(t)$ is:

$$\frac{d\vec{\mathbf{u}}}{dt} = M(t)^{-1} \begin{bmatrix} 0 \\ f(t)/a_2 \end{bmatrix}$$

Example 2: Solve $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$.

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$$\frac{d\vec{\mathbf{u}}}{dt} = M(t)^{-1}\begin{bmatrix} 0 \\ f(t)/a_2 \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{2}t} & e^t \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix}$$

$$= \frac{1}{\frac{3}{2}e^{\frac{1}{2}t}} \begin{bmatrix} e^t & -e^t \\ \frac{1}{2}e^{-\frac{1}{2}t} & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3e^{\frac{3}{2}t}}{2(1+e^t)} \end{bmatrix} = \begin{bmatrix} -\frac{e^{2t}}{1+e^t} \\ \frac{e^{\frac{1}{2}t}}{1+e^t} \end{bmatrix}.$$

Example 2 (continued): $2y'' - y' - y = \dfrac{3e^{\frac{3}{2}t}}{1+e^t}$

- Integrate:

$$
\begin{aligned}
u_1(t) &= \int -\tfrac{e^{2t}}{1+e^t}dt \qquad \text{substitute } p = e^t, dp = e^t dt \\
&= \int -\tfrac{pdp}{1+p} = \int (-1 + \tfrac{1}{1+p})dp = -p + \ln|1+p| + C_1 \\
&= -e^t + \ln|1+e^t| + C_1 \\
&= -e^t + \ln(1+e^t) + C_1,
\end{aligned}
$$

$$
\begin{aligned}
u_2(t) &= \int \tfrac{e^{\frac{1}{2}t}}{1+e^t}dt \qquad \text{substitute } q = e^{t/2}, dq = \tfrac{1}{2}e^{t/2}dt \\
&= \int \tfrac{2dq}{1+q^2} = 2\arctan q + C_2 \\
&= 2\arctan(e^{t/2}) + C_2.
\end{aligned}
$$

Example 2 (continued): $2y'' - y' - y = \frac{3e^{\frac{3}{2}t}}{1+e^t}$

- Integrate:

$$\begin{aligned} u_1(t) &= \int -\tfrac{e^{2t}}{1+e^t}dt \qquad \text{substitute } p = e^t, dp = e^t dt \\ &= \int -\tfrac{pdp}{1+p} = \int (-1 + \tfrac{1}{1+p})dp = -p + \ln|1+p| + C_1 \\ &= -e^t + \ln|1+e^t| + C_1 \\ &= -e^t + \ln(1+e^t) + C_1, \end{aligned}$$

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- Finally, the solutions $y(t)$ of the nonhomog diff eq are obtained:

$$\begin{aligned} y(t) &= y_1u_1 + y_2u_2 \\ &= e^{-\frac{1}{2}t}\left[-e^t + \ln(1+e^t) + C_1\right] + e^t\left[2\arctan(e^{t/2}) + C_2\right] \\ &= -e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\ln(1+e^t) + 2e^t\arctan(e^{\frac{1}{2}t}) + C_1e^{-\frac{1}{2}t} + C_2e^t. \end{aligned}$$

Example Solve $(1+t)y'' + (-1-2t)y' + ty = \frac{e^t}{1+t}$,

given that complementary solutions are $y_c(t) = c_1 e^t + c_2 e^t \ln|1+t|$.

Example Solve $(1+t)y'' + (-1-2t)y' + ty = \frac{e^t}{1+t}$,

given that complementary solutions are $y_c(t) = c_1 e^t + c_2 e^t \ln|1+t|$.

Solution • $y_c(t) = C_1 y_1(t) + C_2 y_2(t)$ with $y_1(t) = e^t$, $y_2(t) = e^t \ln|1+t|$

• A fund. matrix $M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

• Set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = M(t)\vec{u}(t)$.

The Diff Eq for $\vec{u}(t)$ is: $\frac{d\vec{u}}{dt} = M(t)^{-1} \begin{bmatrix} 0 \\ f(t)/a_2(t) \end{bmatrix}$

• Integrate:

• $y(t) = y_1(t)u_1(t) + y_2(t)u_2(t)$

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• A fund. matrix $M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^t & e^t \ln|1+t| \\ e^t & e^t \ln|1+t| + \frac{e^t}{1+t} \end{bmatrix}$.

• Set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = M(t)\vec{u}(t)$.

The Diff Eq for $\vec{u}(t)$ is: $\frac{d\vec{u}}{dt} = M(t)^{-1} \begin{bmatrix} 0 \\ f(t)/a_2(t) \end{bmatrix} = M(t)^{-1} \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix}$.

$$\frac{d\vec{u}}{dt} = \frac{1}{\frac{e^{2t}}{1+t}} \begin{bmatrix} e^t \ln|1+t| + \frac{e^t}{1+t} & -e^t \ln|1+t| \\ -e^t & e^t \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^t}{(1+t)^2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\ln|1+t|}{1+t} \\ \frac{1}{1+t} \end{bmatrix}.$$

$\int -\frac{\ln|1+t|}{1+t} dt = \int -s\, ds = -\frac{1}{2}s^2 + C$.
↑ Substitute $s = \ln|1+t|$

$\int \frac{1}{1+t} dt = \ln|1+t| + C$.

• Integrate: $\vec{u}(t) = \begin{bmatrix} -\frac{1}{2}(\ln|1+t|)^2 + C_1 \\ \ln|1+t| + C_2 \end{bmatrix}$

• $y(t) = y_1(t)u_1(t) + y_2(t)u_2(t) = e^t\left\{-\frac{1}{2}(\ln|1+t|)^2 + C_1\right\} + e^t \ln|1+t| \left\{\ln|1+t| + C_2\right\}$

$$= \frac{1}{2}e^t(\ln|1+t|)^2 + C_1 e^t + C_2 e^t \ln|1+t|.$$

Example Solve $4y''' + 9y' = \sec \frac{3t}{2}$

Example Solve $4y''' + 9y' = \sec\frac{3t}{2}$

Solution $y(t) = y_p(t) + y_c(t)$.

Step 1 Find $y_c(t)$ and a fund. matrix $M(t)$.

- $4\lambda^3 + 9\lambda = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{3}{2}i,\ \lambda_3 = -\frac{3}{2}i \Rightarrow y_c(t) = C_1 + C_2\cos\frac{3t}{2} + C_3\sin\frac{3t}{2}$

  $y_1 = 1,\ y_2 = \cos\frac{3t}{2},\ y_3 = \sin\frac{3t}{2}$

- A fund. matrix $M(t) =$

Step 2. Set $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = M(t)\vec{u}(t)$. The Diff Eq for $\vec{u}(t)$ becomes:

$$\frac{d\vec{u}}{dt} = M(t)^{-1}\begin{bmatrix} 0 \\ 0 \\ \frac{\sec\frac{3t}{2}}{4} \end{bmatrix}:$$

Step 3 Integrate $\Rightarrow \vec{u}_p(t) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \Rightarrow y_p(t) = y_1 u_{1p} + y_2 u_{2p} + y_3 u_{3p}$

Step 4 Gen. Sol's of the Nonhomog. Eq:

$y(t) = y_p(t) + y_c(t)$:

Example Solve $4y''' + 9y' = \sec\frac{3t}{2}$

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Step 1 Find $y_c(t)$ and a fund. matrix $M(t)$.

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$y_1 = 1,\ y_2 = \cos\frac{3t}{2},\ y_3 = \sin\frac{3t}{2}$

- A fund. matrix $M(t) = \begin{bmatrix} 1 & \cos\frac{3t}{2} & \sin\frac{3t}{2} \\ 0 & -\frac{3}{2}\sin\frac{3t}{2} & \frac{3}{2}\cos\frac{3t}{2} \\ 0 & -\frac{9}{4}\cos\frac{3t}{2} & -\frac{9}{4}\sin\frac{3t}{2} \end{bmatrix}$

Step 2. Set $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = M(t)\vec{u}(t)$. The Diff Eq for $\vec{u}(t)$ becomes:

$$\frac{d\vec{u}}{dt} = M(t)^{-1}\begin{bmatrix} 0 \\ 0 \\ \frac{\sec\frac{3t}{2}}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{4}{9} \\ 0 & -\frac{2}{3}\sin\frac{3t}{2} & -\frac{4}{9}\cos\frac{3t}{2} \\ 0 & \frac{2}{3}\cos\frac{3t}{2} & -\frac{4}{9}\sin\frac{3t}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{4}\sec\frac{3t}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{9}\sec\frac{3t}{2} \\ -\frac{1}{9} \\ -\frac{1}{9}\tan\frac{3t}{2} \end{bmatrix}$$

Step 3 Integrate $\Rightarrow$ $\vec{u}_p(t) = \begin{bmatrix} \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| \\ -\frac{1}{9}t \\ \frac{2}{27}\ln\left|\cos\frac{3t}{2}\right| \end{bmatrix} \Rightarrow y_p(t) = y_1 u_{1p} + y_2 u_{2p} + y_3 u_{3p}$

Step 4 Gen. Sol's of the Nonhomog. Eq:

$$y(t) = y_p(t) + y_c(t) = \frac{2}{27}\ln\left|\sec\frac{3t}{2} + \tan\frac{3t}{2}\right| - \frac{1}{9}t\cos\frac{3t}{2} + \frac{2}{27}\sin\frac{3t}{2}\ln\left|\cos\frac{3t}{2}\right| + C_1 + C_2\cos\frac{3t}{2} + C_3\sin\frac{3t}{2}.$$