Oscillations driven by external forces

Frequency response

Resonance

Forced Vibration in Spring-Mass

Assume that there are three forces only:

- The restoring force exerted by the spring: $-ky$;
- The damping force: $-\gamma \frac{dy}{dt}$.
- The external force: $F(t)$.

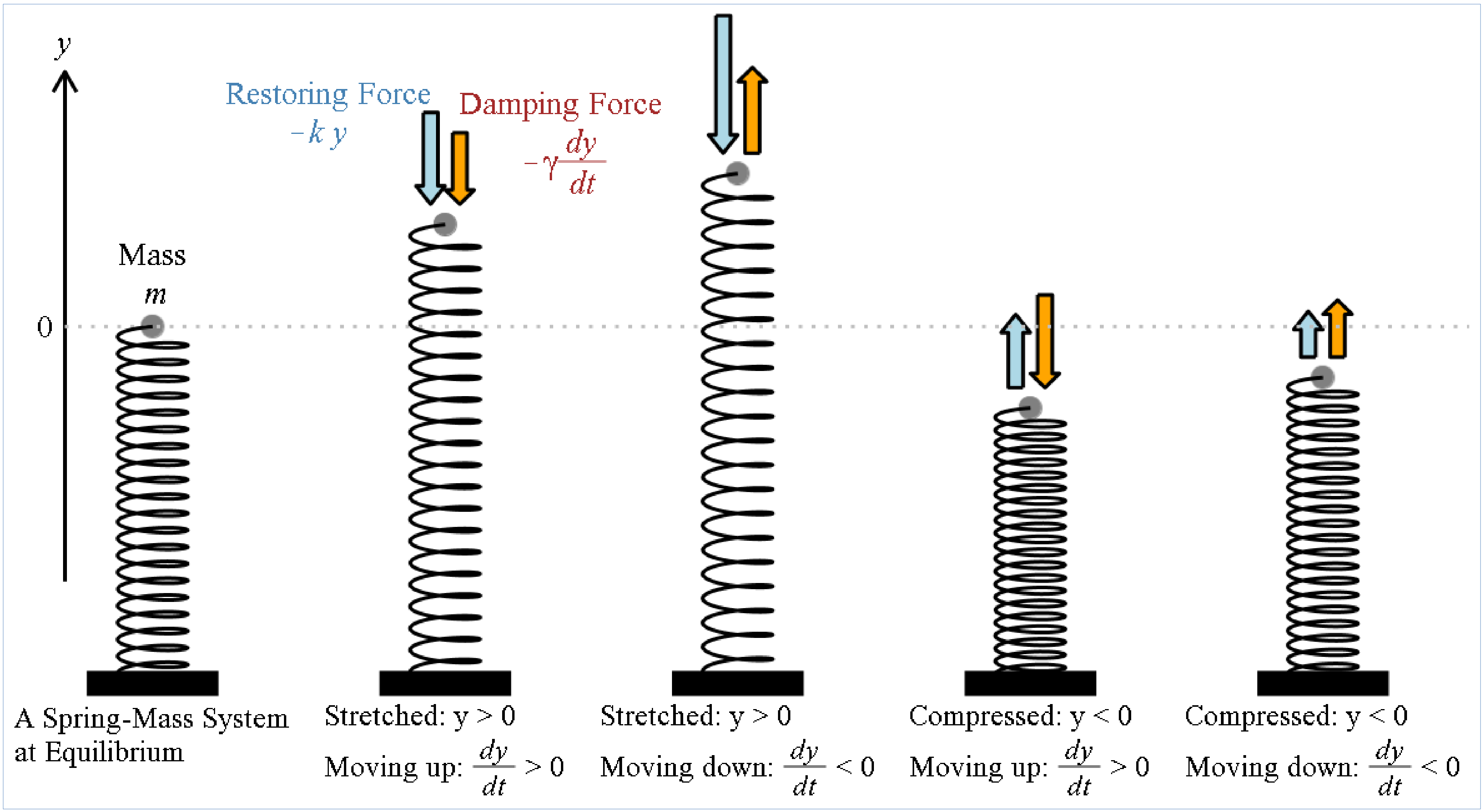
Here, k is the spring constant and γ is the damping coefficient.

The Derivation of the Diff Equation

$$\text{(mass)(acceleration)} = \text{(the net force on the mass)},$$

$$m\frac{d^2y}{dt^2} = -ky - \gamma\frac{dy}{dt} + F(t),$$

$$\boxed{m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F(t)}$$

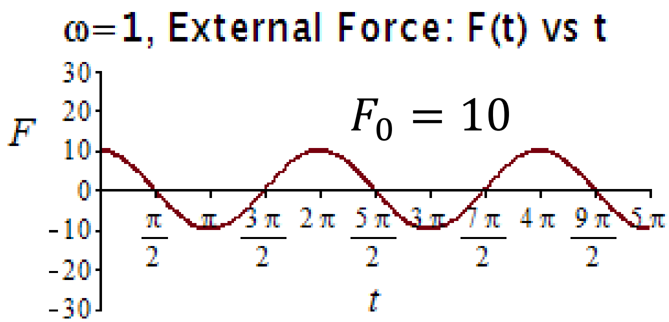


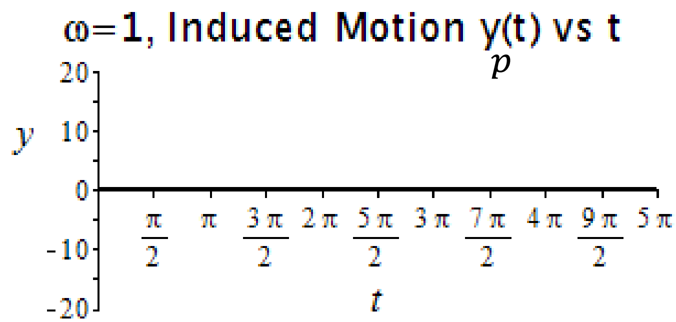
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

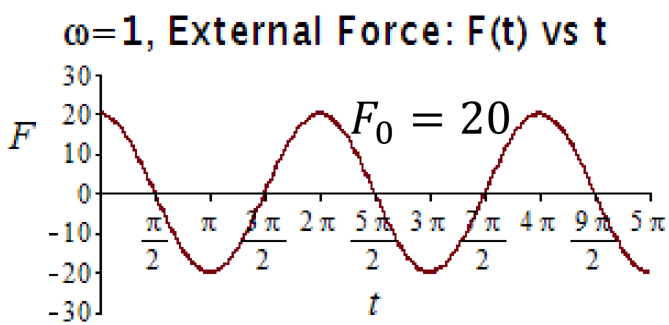
$$F(t) = F_0\cos(\omega t)$$

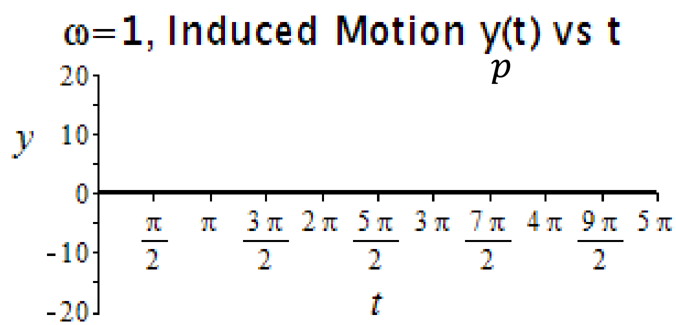
$$y_p(t) = A\cos(\omega t)$$

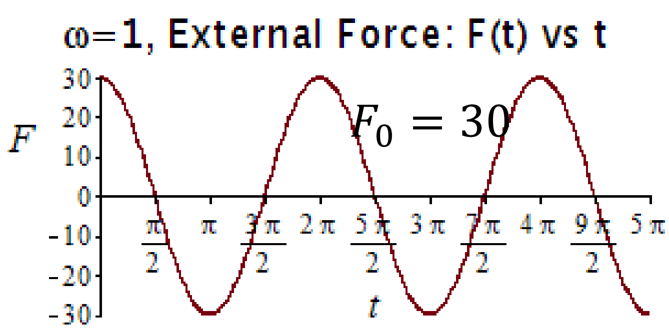
Question:
How does A depend on F_0?

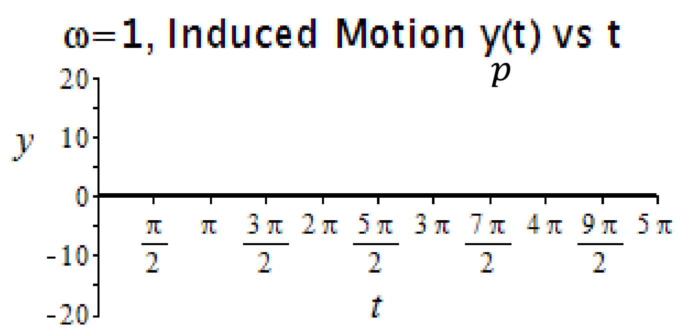










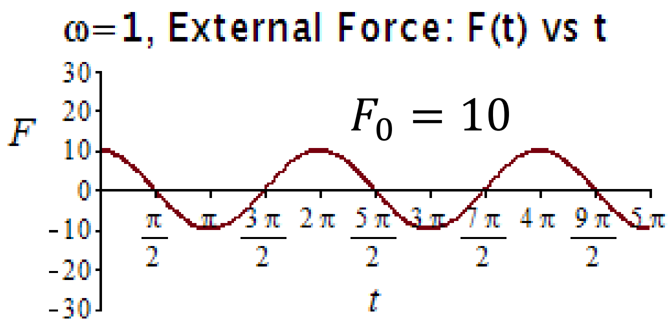


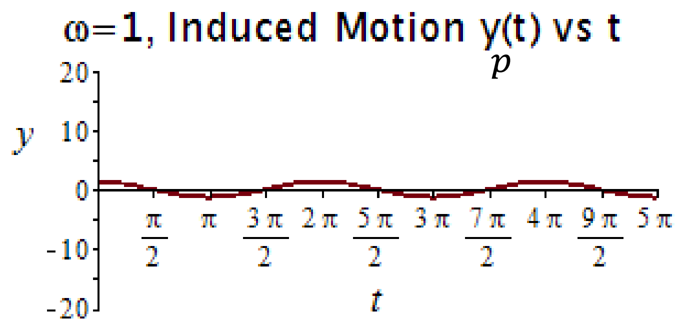
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

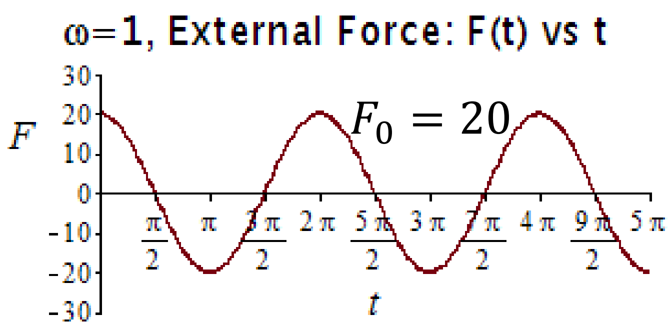
$$F(t) = F_0\cos(\omega t) \qquad y_p(t) = A\cos(\omega t)$$

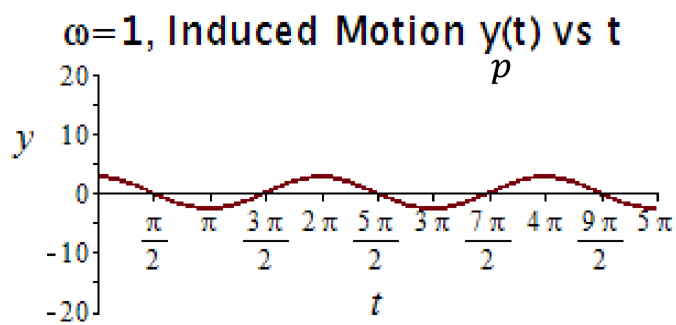
Question:
How does A depend on F_0?

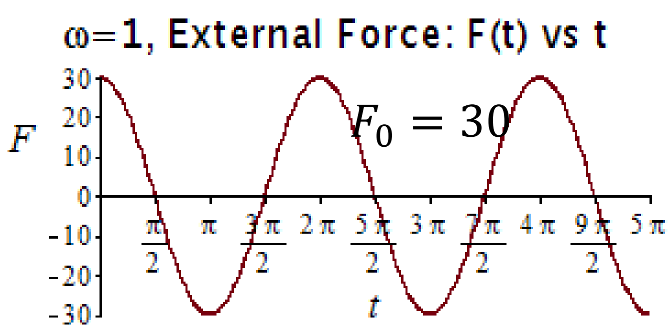
Answer:
$A \propto F_0$.
A larger force induces
a larger motion.

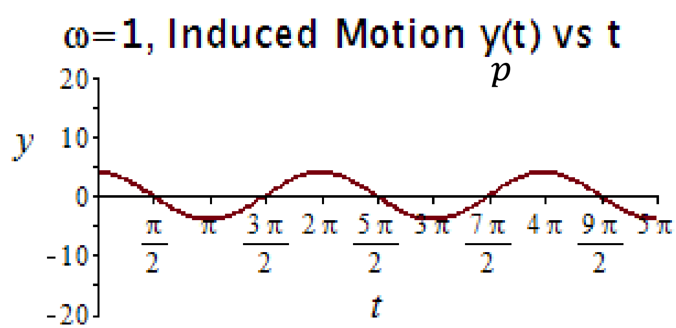








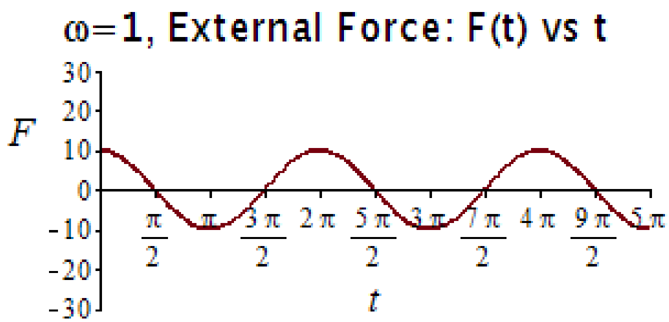


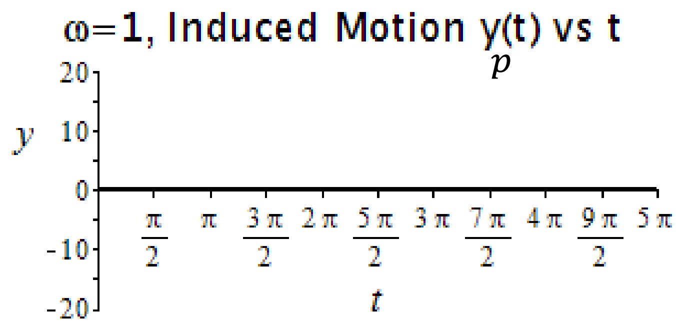


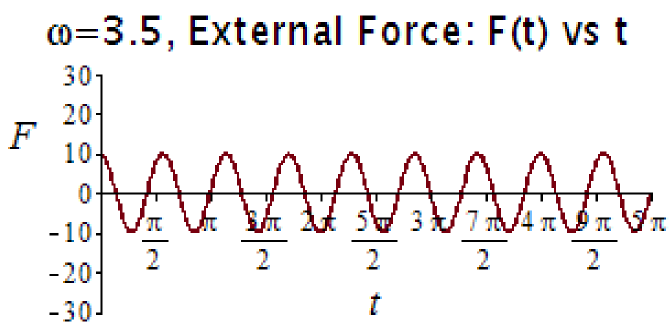
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

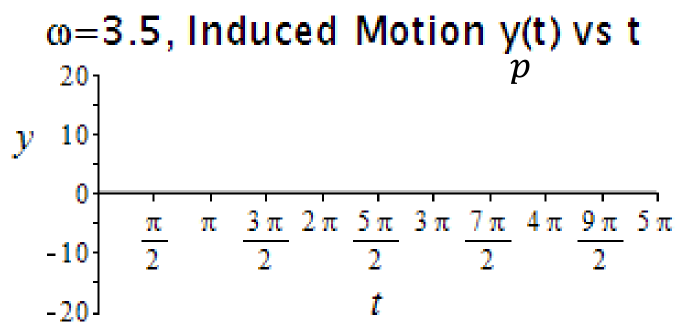
$$F(t) = 10\cos(\omega t) \qquad y_p(t) = A\cos(\omega t)$$

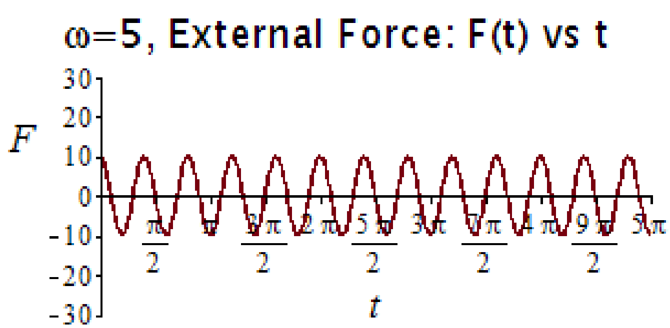
Question:
How does A depend on ω?

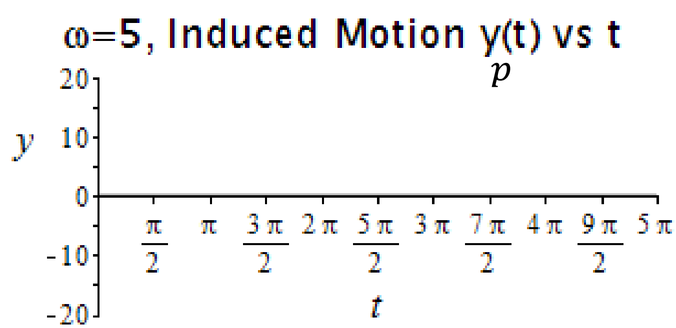












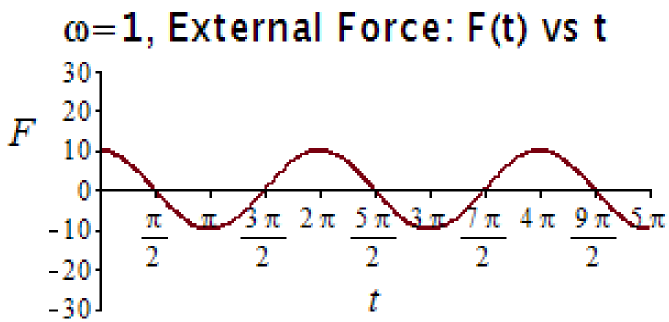
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

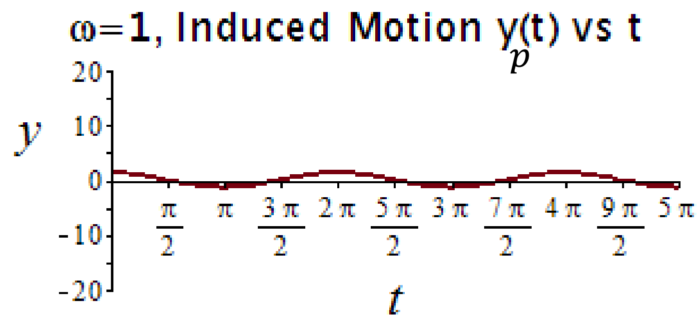
Frequency Response

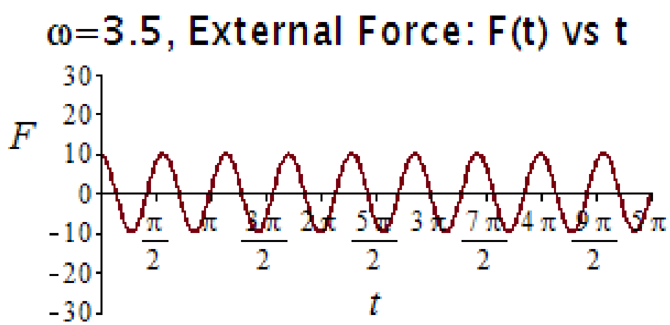
Question:
How does A depend on ω?

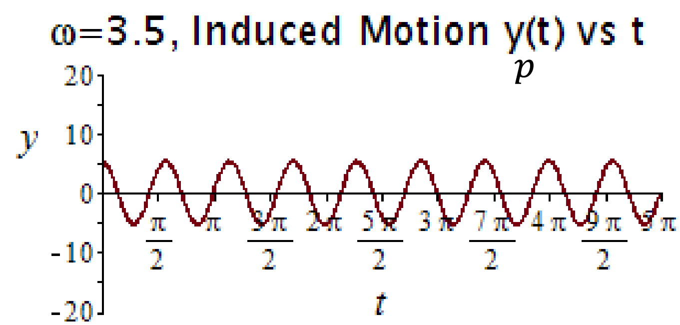
$$F(t) = 10\cos(\omega t)$$

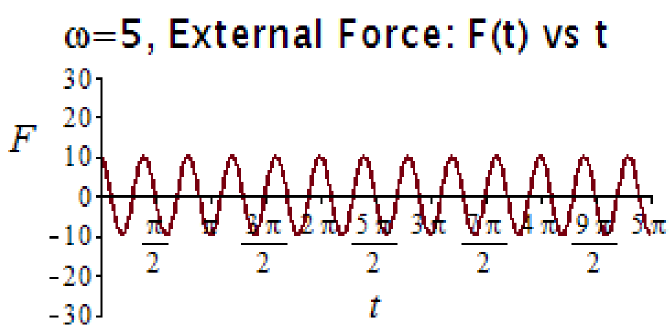
$$y_p(t) = A\cos(\omega t)$$

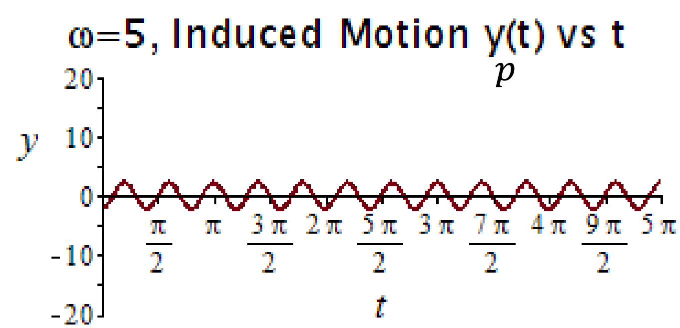










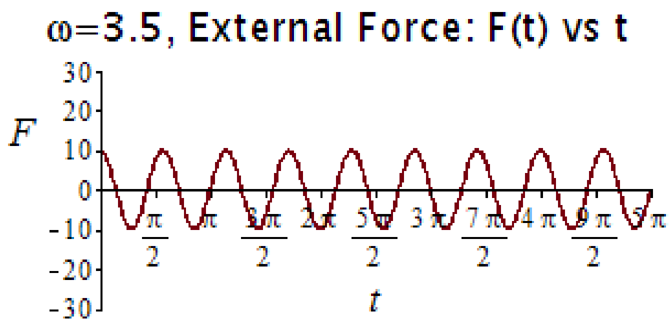


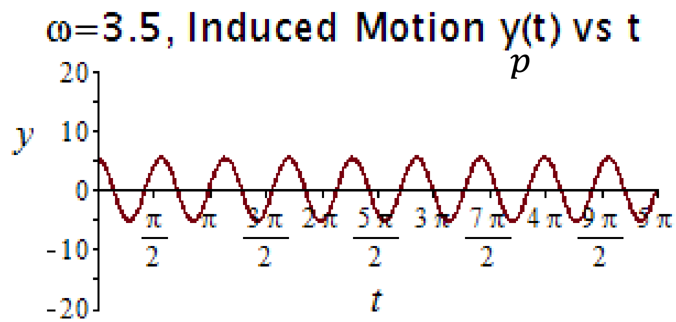
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

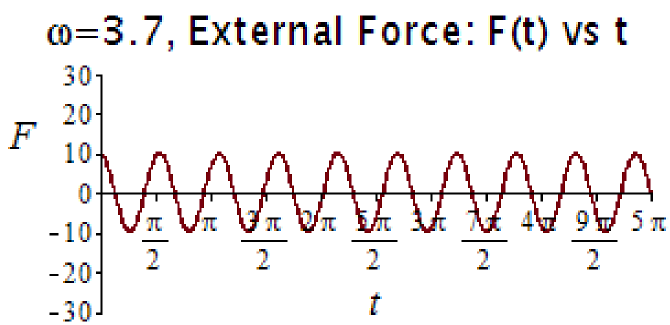
Frequency Response

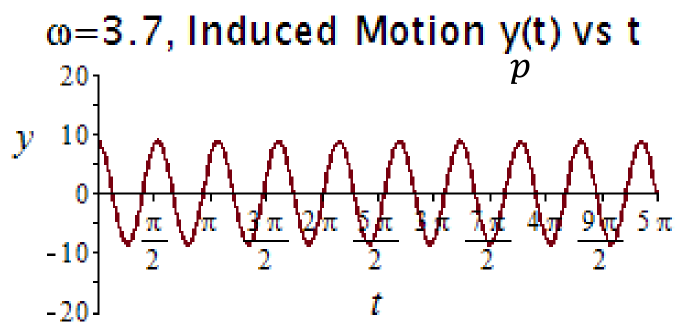
Question:
How does A depend on ω?

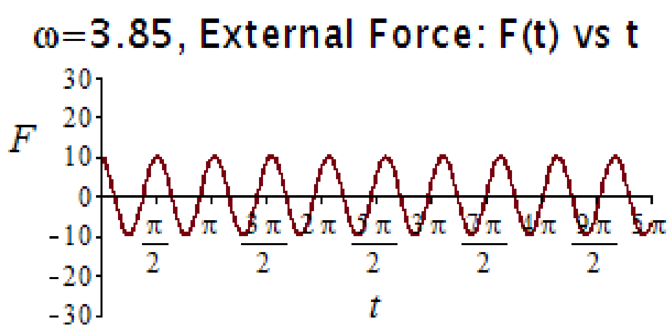
$$F(t) = 10\cos(\omega t) \qquad y_p(t) = A\cos(\omega t)$$

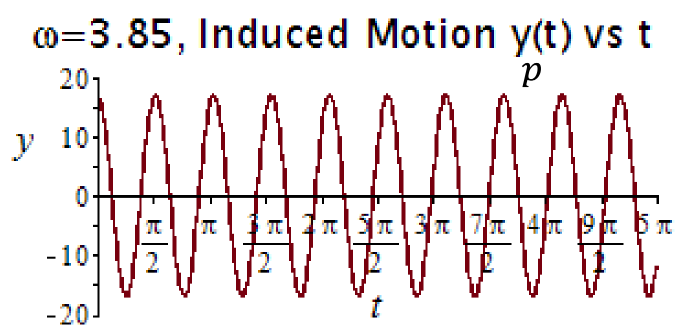












The Natural Frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4$$

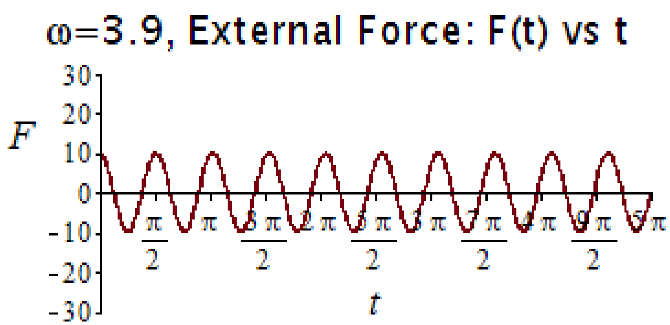
$$\frac{1}{2}\frac{d^2y}{dt^2} + 8y = F(t)$$

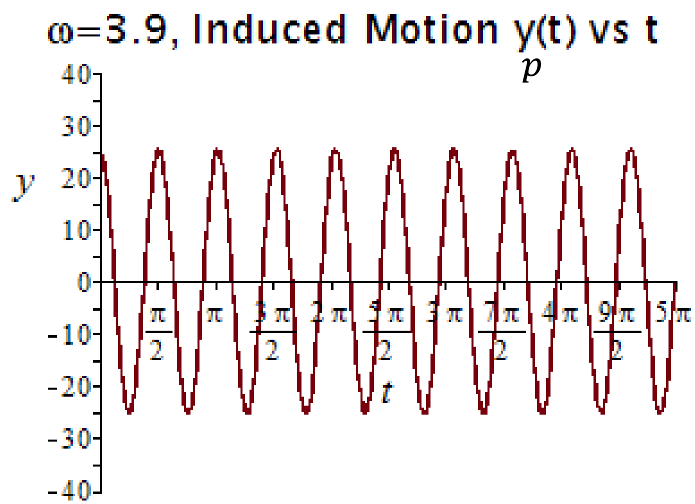
Frequency Response

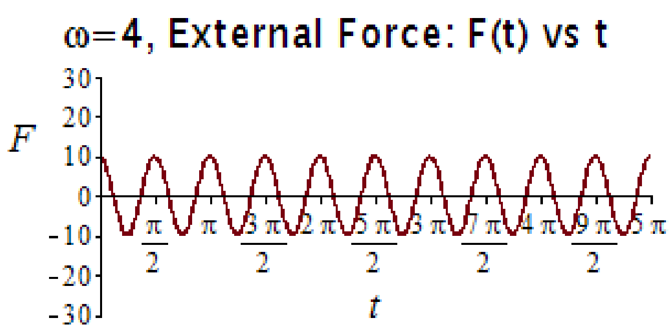
Question:
How does A depend on ω?

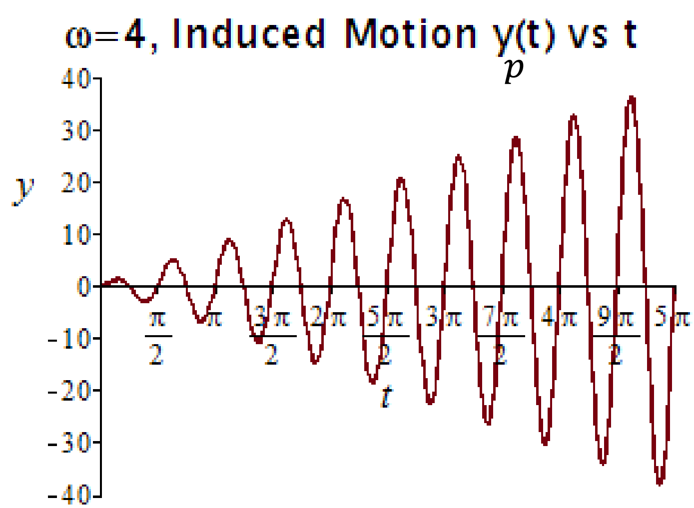
$$F(t) = 10\cos(\omega t)$$

$$y_p(t) = A\cos(\omega t)$$









Resonance

The Natural Frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4$$

Un-Damped Spring-Mass System

$$m\frac{d^2y}{dt^2} + ky = F(t)$$

$F(t) = F_0\cos(\omega t)$	$y_p(t) = \frac{F_0}{m}G(i\omega)\cos(\omega t)$
$F(t) = F_0\sin(\omega t)$	$y_p(t) = \frac{F_0}{m}G(i\omega)\sin(\omega t)$
$F(t) = F_0 e^{i\omega t}$	$y_p(t) = \frac{F_0}{m}G(i\omega)e^{i\omega t}$

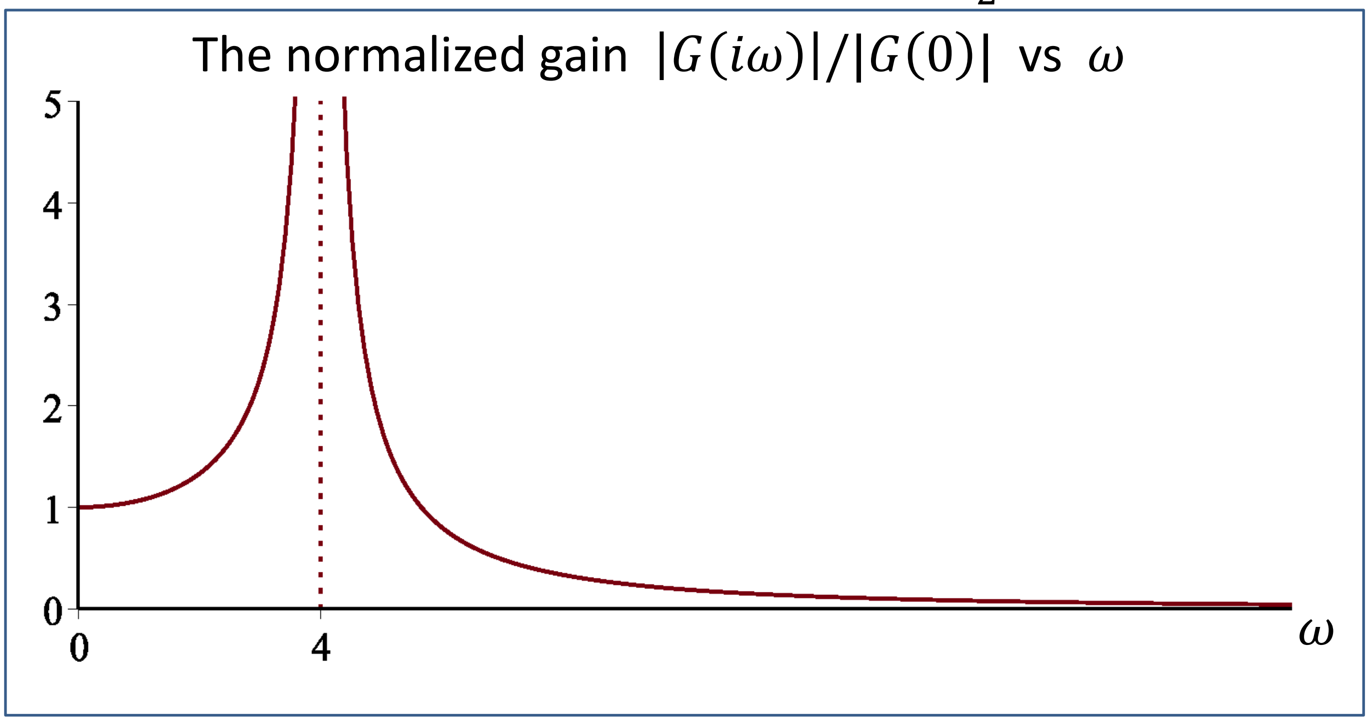
The Natural Frequency:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

The Frequency Response:

$$G(i\omega) = \frac{1}{(i\omega)^2+\omega_0^2} = \frac{1}{-\omega^2+\omega_0^2}$$

The Frequency Response Curve when $m = \frac{1}{2}, \gamma = 0, k = 8$



At the natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4,$$

the gain diverges.

Damped Spring-Mass System

$$m\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+ky=F(t)$$

$F(t)=F_0e^{i\omega t}$	$y_p(t)=\frac{F_0}{m}G(i\omega)e^{i\omega t}$
$F(t)=F_0\cos(\omega t)$	$y_p(t)=Re\left[\frac{F_0}{m}G(i\omega)e^{i\omega t}\right]$
$F(t)=F_0\sin(\omega t)$	$y_p(t)=Im\left[\frac{F_0}{m}G(i\omega)e^{i\omega t}\right]$

The Natural Frequency:

$$\omega_0=\sqrt{\frac{k}{m}}$$

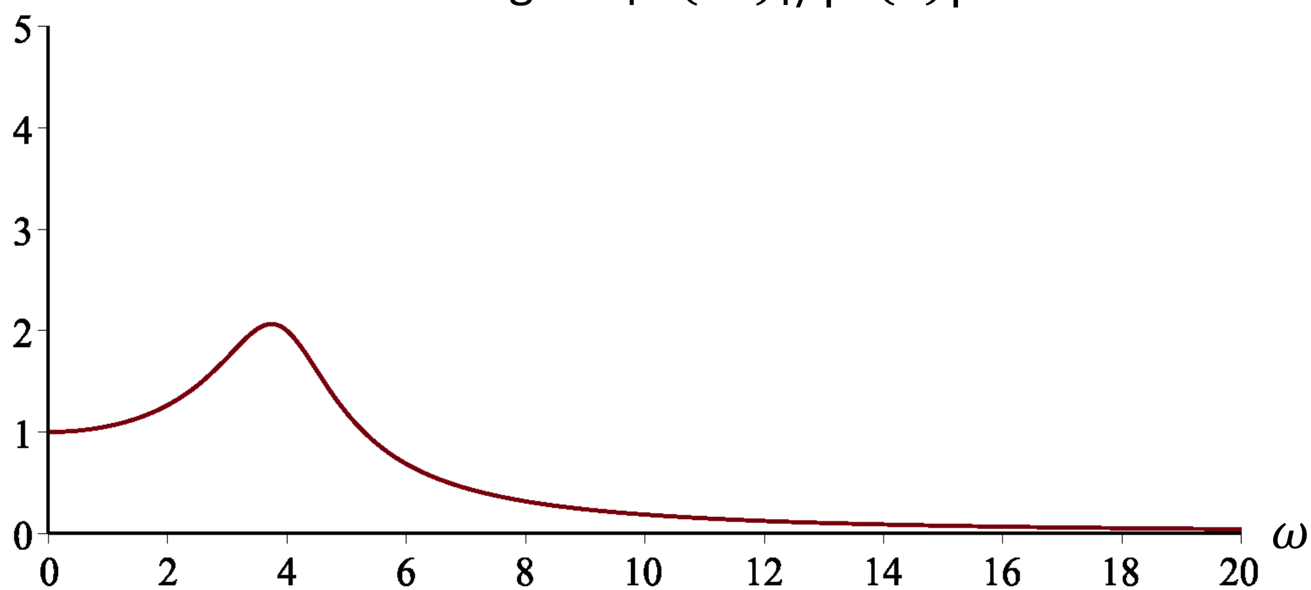
The Frequency Response:

$$G(i\omega)=\frac{1}{(i\omega)^2+\frac{\gamma}{m}(i\omega)+\omega_0^2}$$

$$=\frac{1}{-\omega^2+\omega_0^2+\frac{\gamma\omega}{m}i}$$

The Frequency Response Curve when $m=\frac{1}{2},\gamma=1,k=8$

The normalized gain $|G(i\omega)|/|G(0)|$ vs ω



The natural frequency:

$$\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{8}{1/2}}=4.$$

The maximal gain is at:

$$\omega_{max}=\sqrt{\omega_0^2-\frac{\gamma^2}{2m^2}}=\sqrt{16-\frac{1}{2(1/2)^2}}=\sqrt{14}.$$

Damped Spring-Mass System

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F(t)$$

$F(t) = F_0 e^{i\omega t}$	$y_p(t) = \frac{F_0}{m} G(i\omega) e^{i\omega t}$
$F(t) = F_0 \cos(\omega t)$	$y_p(t) = Re\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$
$F(t) = F_0 \sin(\omega t)$	$y_p(t) = Im\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$

The Natural Frequency:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

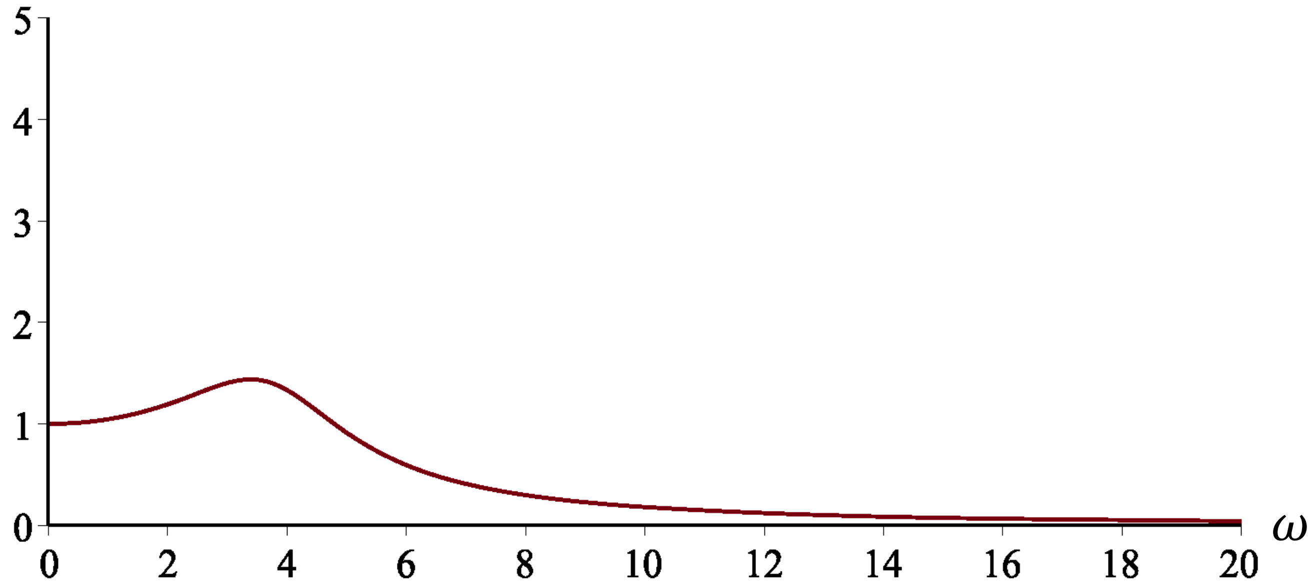
The Frequency Response:

$$G(i\omega) = \frac{1}{(i\omega)^2 + \frac{\gamma}{m}(i\omega) + \omega_0^2}$$

$$= \frac{1}{-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i}$$

The Frequency Response Curve when $m = \frac{1}{2}, \gamma = 1.5, k = 8$

The normalized gain $|G(i\omega)|/|G(0)|$ vs ω



The natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4.$$

The maximal gain is at:

$$\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}} = \sqrt{16 - \frac{1.5}{2(1/2)^2}} = \sqrt{11.5}.$$

Damped Spring-Mass System

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F(t)$$

$F(t) = F_0 e^{i\omega t}$	$y_p(t) = \frac{F_0}{m} G(i\omega) e^{i\omega t}$
$F(t) = F_0 \cos(\omega t)$	$y_p(t) = Re\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$
$F(t) = F_0 \sin(\omega t)$	$y_p(t) = Im\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$

The Natural Frequency:

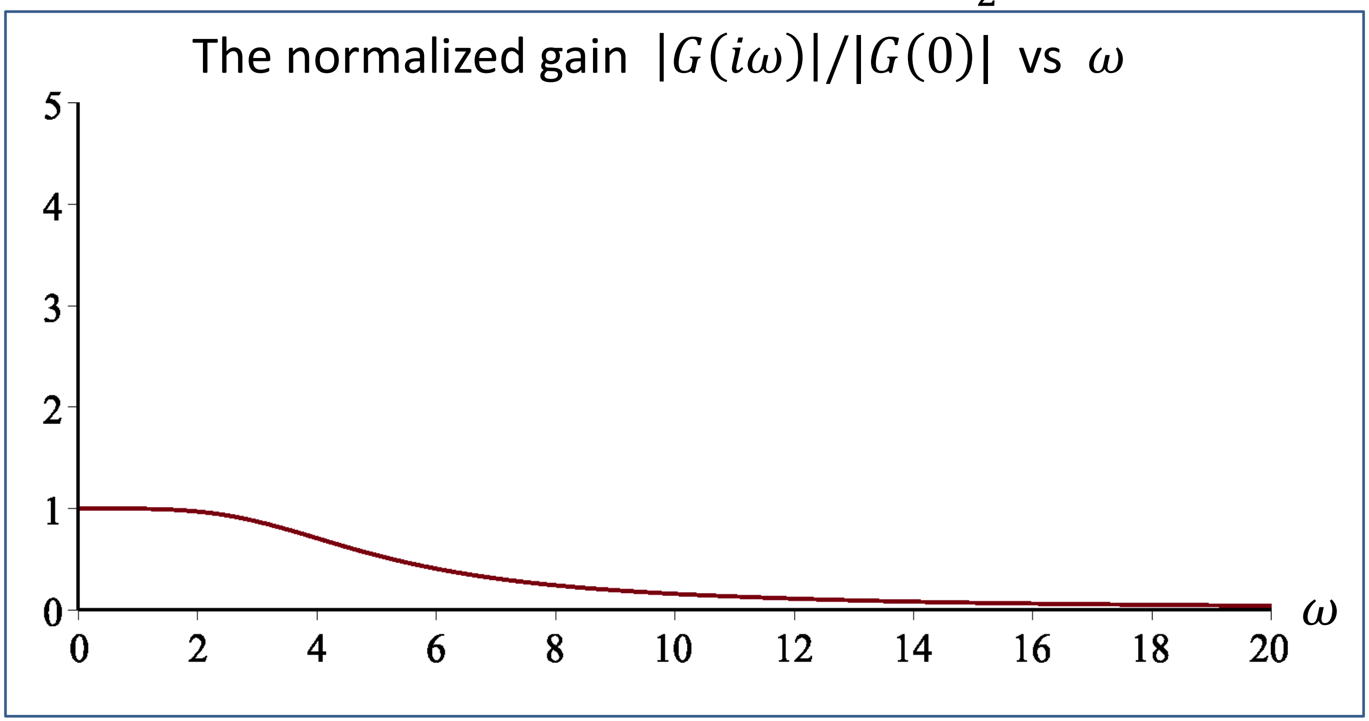
$$\omega_0 = \sqrt{\frac{k}{m}}$$

The Frequency Response:

$$G(i\omega) = \frac{1}{(i\omega)^2 + \frac{\gamma}{m}(i\omega) + \omega_0^2}$$

$$= \frac{1}{-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i}$$

The Frequency Response Curve when $m = \frac{1}{2}, \gamma = \sqrt{8}, k = 8$



The natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4.$$

The maximal gain is at:

$$\omega_{max} = 0.$$

Damped Spring-Mass System

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F(t)$$

$F(t) = F_0 e^{i\omega t}$	$y_p(t) = \frac{F_0}{m} G(i\omega) e^{i\omega t}$
$F(t) = F_0 \cos(\omega t)$	$y_p(t) = Re\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$
$F(t) = F_0 \sin(\omega t)$	$y_p(t) = Im\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right]$

The Natural Frequency:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

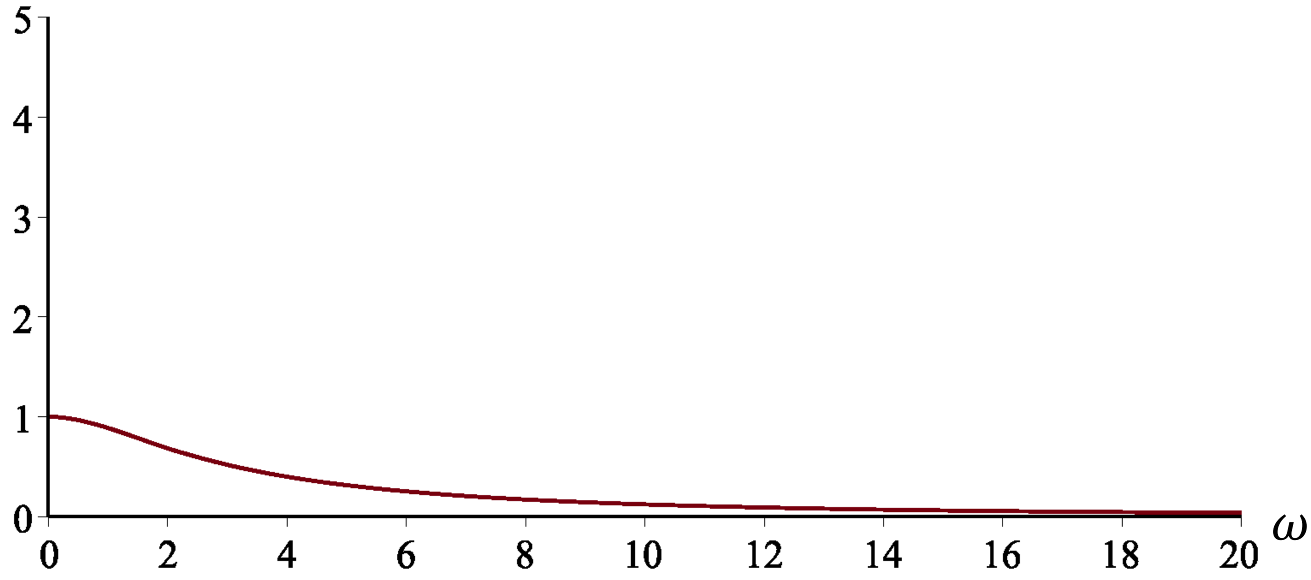
The Frequency Response:

$$G(i\omega) = \frac{1}{(i\omega)^2 + \frac{\gamma}{m}(i\omega) + \omega_0^2}$$

$$= \frac{1}{-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i}$$

The Frequency Response Curve when $m = \frac{1}{2}, \gamma = 5, k = 8$

The normalized gain $|G(i\omega)|/|G(0)|$ vs ω



The natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4.$$

The maximal gain is at:

$$\omega_{max} = 0.$$

The Frequency Response:

$$G(i\omega) = \frac{1}{(i\omega)^2 + \frac{\gamma}{m}(i\omega) + \omega_0^2}$$

$$= \frac{1}{-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i}$$

The Normalized Gain:

$$\left|\frac{G(i\omega)}{G(0)}\right| = \frac{\omega_0^2}{\left|-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i\right|}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\gamma^2}{mk} - 2\right)\left(\frac{\omega}{\omega_0}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^4}}$$

The Frequency Response Curves when $m = \frac{1}{2}, k = 8$,

for $\gamma = 0,\ 1,\ 1.5,\ \sqrt{2mk} = \sqrt{8},\ 5$.

The normalized gain $|G(i\omega)|/|G(0)|$ vs ω