Second Order Homogeneous Linear Differential Equations: the method of reduction of order

Xu-Yan Chen

- **Second Order Homogeneous Linear Differential Equations:**

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

- **General solution structure:**

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

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I will get you **all** solutions.

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  Or, equivalently, $\quad y(t) = C_1 \dfrac{t}{2}\ln(1+2t) + C_2 t.$

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$$4(e^{\frac{3}{2}t}u'' + 3e^{\frac{3}{2}t}u' + \frac{9}{4}e^{\frac{3}{2}t}u) - 12(e^{\frac{3}{2}t}u' + \frac{3}{2}e^{\frac{3}{2}t}u) + 9e^{\frac{3}{2}t}u = 0,$$
$$4e^{\frac{3}{2}t}u'' = 0,$$

$$(*)_u \qquad u'' = 0.$$

- Integrate once: $\quad u' = C_1$
- Integrate again: $\quad u(t) = C_1 t + C_2$
- Finally, get y from u: $\quad y(t) = y_1(t)u(t) = e^{\frac{3}{2}t}u(t)$.

$$y(t) = C_1\, te^{\frac{3}{2}t} + C_2 e^{\frac{3}{2}t}.$$