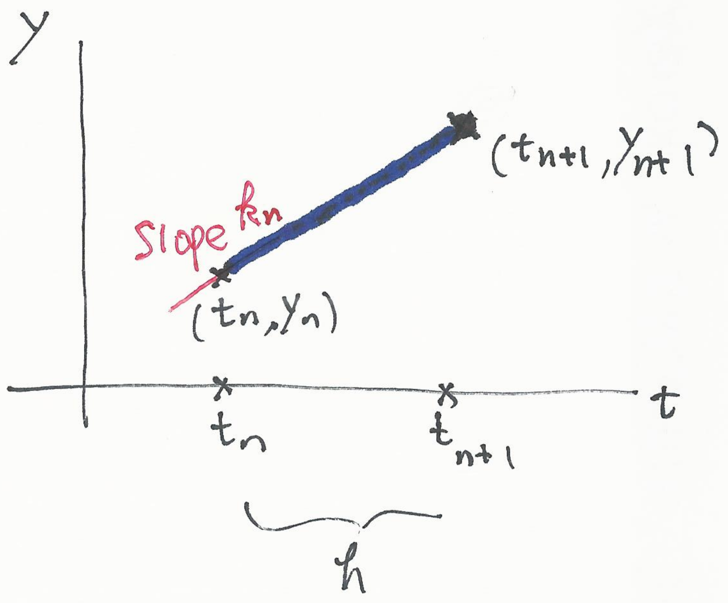
Euler's Method

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

$$\boxed{t_n, y_n} \Longrightarrow \boxed{t_{n+1}, y_{n+1}}$$

- $t_{n+1} = t_n + h$
- $y_{n+1} = y_n + h k_n$

where $k_n = f(t_n, y_n)$



Local Truncation Error $O(h^2)$

Global Truncation Error $O(h)$

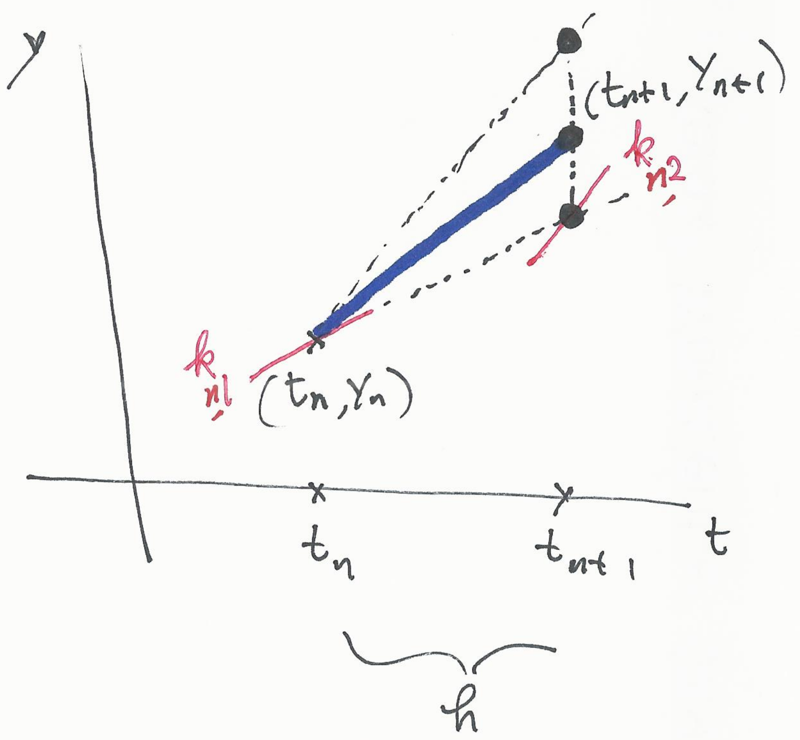
Improved Euler Method

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

$$\boxed{t_n, y_n} \Longrightarrow \boxed{t_{n+1}, y_{n+1}}$$

- $t_{n+1} = t_n + h$
- $y_{n+1} = y_n + h \dfrac{k_{n,1} + k_{n,2}}{2}$

where
$$\begin{cases} k_{n,1} = f(t_n, y_n) \\ k_{n,2} = f(t_n + h, y_n + h k_{n,1}) \end{cases}$$



Use the Average Slope

$$\frac{k_{n,1} + k_{n2}}{2}$$

Local Truncation Error

$O(h^3)$

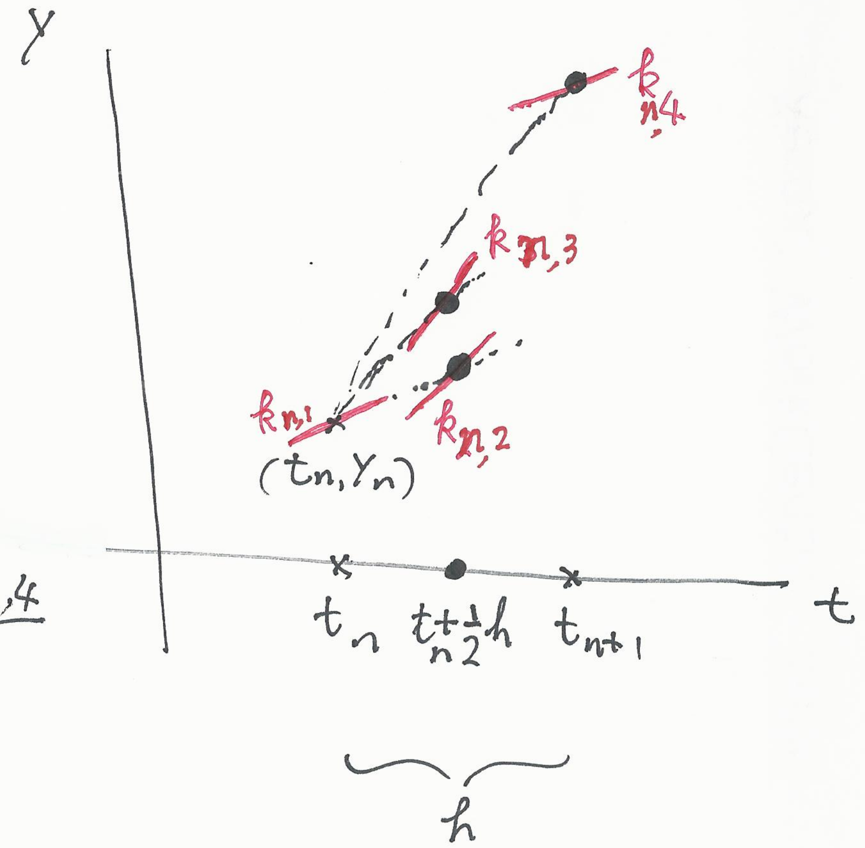
Global Truncation Error

$O(h^2)$

Runge-Kutta Method

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 .$$

$\boxed{t_n, y_n} \Rightarrow \boxed{t_{n+1}, y_{n+1}}$



- $t_{n+1} = t_n + h$
- $y_{n+1} = y_n + h \dfrac{k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}}{6}$

where $k_{n,1} = f(t_n, y_n)$

$$\begin{cases} k_{n,2} = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2} k_{n,1}) \\ k_{n,3} = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2} k_{n,2}) \\ k_{n,4} = f(t_n + h, y_n + h k_{n,3}) \end{cases}$$

Use the Average Slope:

$$\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}$$

Local Truncation Error $O(h^5)$

Global Truncation Error $O(h^4)$

Example: $\dfrac{dy}{dt} = 1 - t + 4y, \qquad y(0) = 1$

Improved Euler Method
with $h = 0.2$ (Step size)
$f(t, y) = 1 - t + 4y$

$$\textbf{Iteration}: \quad t_{n+1} = t_n + h, \qquad y_{n+1} = y_n + h\frac{k_{n,1} + k_{n,2}}{2}$$

- **Start point:** $t_0 = 0, y_0 = 1$

$k_{0,1} = f(t_0, y_0) = f(0, 1) = 1 - 0 + 4(1) = 5$
$k_{0,2} = f\left(t_0 + h, y_0 + hk_{0,1}\right) = f(0.2, 2) = 1 - 0.2 + 4(2) = 8.8$

$$t_1 = t_0 + h = 0.2$$
$$y_1 = y_0 + h\frac{k_{0,1} + k_{0,2}}{2} = 1 + (0.2)\frac{5 + 8.8}{2} = 2.38$$

- **Next point:** $t_1 = 0.2, y_1 = 2.38$

$k_{1,1} = f(t_1, y_1) = f(0.2, 2.38) = 1 - 0.2 + 4(2.38) = 10.32$
$k_{1,2} = f\left(t_1 + h, y_1 + hk_{1,1}\right) = f(0.4, 4.444) = 1 - 0.4 + 4(4.444) = 18.376$

$$t_2 = t_1 + h = 0.4$$
$$y_2 = y_1 + h\frac{k_{1,1} + k_{1,2}}{2} = 2.38 + (0.2)\frac{10.32 + 18.376}{2} = 5.2496$$

- **Next point:** $t_2 = 0.4, y_2 = 5.2496$

Keep going ....

Example: $\dfrac{dy}{dt} = 1 - t + 4y, y(0) = 1$ **Runge-Kutta Method** with $h = 0.2$ (Step size)

$f(t, y) = 1 - t + 4y$

Iteration: $t_{n+1} = t_n + h, \qquad y_{n+1} = y_n + \frac{h}{6}(k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4})$

- **Start point:** $t_0 = 0, y_0 = 1$

$$k_{0,1} = f(t_0, y_0) = f(0, 1) = 1 - 0 + 4(1) = 5$$
$$k_{0,2} = f\left(t_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}hk_{0,1}\right) = f(0.1, 1.5) = 1 - 0.1 + 4(1.5) = 6.9$$
$$k_{0,3} = f\left(t_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}hk_{0,2}\right) = f(0.1, 1.69) = 1 - 0.1 + 4(1.69) = 7.66$$
$$k_{0,4} = f\left(t_0 + h, y_0 + hk_{0,3}\right) = f(0.2, 2.532) = 1 - 0.2 + 4(2.532) = 10.928$$

$$t_1 = t_0 + h = 0.2$$
$$y_1 = y_0 + \frac{h}{6}(k_{0,1} + 2k_{0,2} + 2k_{0,3} + k_{0,4})$$
$$= 1 + \frac{0.2}{6}[5 + 2(6.9) + 2(7.66) + 10.928] = 2.5016$$

- **Next point:** $t_1 = 0.2, y_1 = 2.5016$

Example: $\frac{dy}{dt} = 1 - t + 4y, y(0) = 1$ **Runge-Kutta Method** with $h = 0.2$ (Step size)

$f(t, y) = 1 - t + 4y$

Iteration: $t_{n+1} = t_n + h, \qquad y_{n+1} = y_n + \frac{h}{6}(k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4})$

- **Next point:** $t_1 = 0.2, y_1 = 2.5016$

$k_{1,1} = f(t_1, y_1) = f(0.2, 2.5016) = 1 - 0.2 + 4(2.5016) = 10.8064$

$k_{1,2} = f\left(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hk_{1,1}\right) = f(0.3, 3.58224) = 15.02896$

$k_{1,3} = f\left(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hk_{1,2}\right) = f(0.3, 4.004496) = 16.717984$

$k_{1,4} = f(t_1 + h, y_1 + hk_{1,3}) = f(0.4, 5.8451968) = 23.9807872$

$t_2 = t_1 + h = 0.4,$

$$y_2 = y_1 + \frac{h}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4})$$

$$= 2.5016 + \frac{0.2}{6}[10.8064 + 2(15.02896) + 2(16.717984) + 23.9807872]$$

$$= 5.77763584$$

- **Next point:** $t_2 = 0.4, y_2 = 5.77763584$

Keep going ………

$$\frac{dy}{dt} = 1 - t + 4y$$
$$y(0) = 1$$

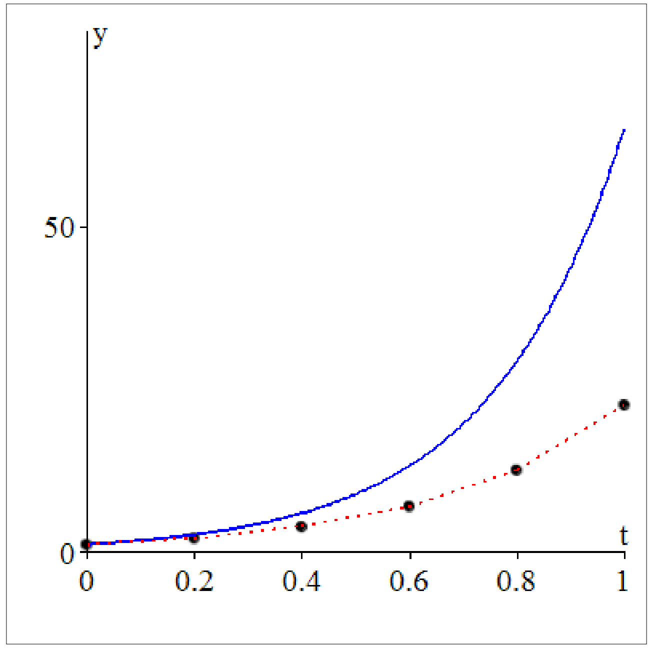
The exact solution

$$y(t) = -\frac{3}{16} + \frac{1}{4}t + \frac{19}{16}e^{4t}$$

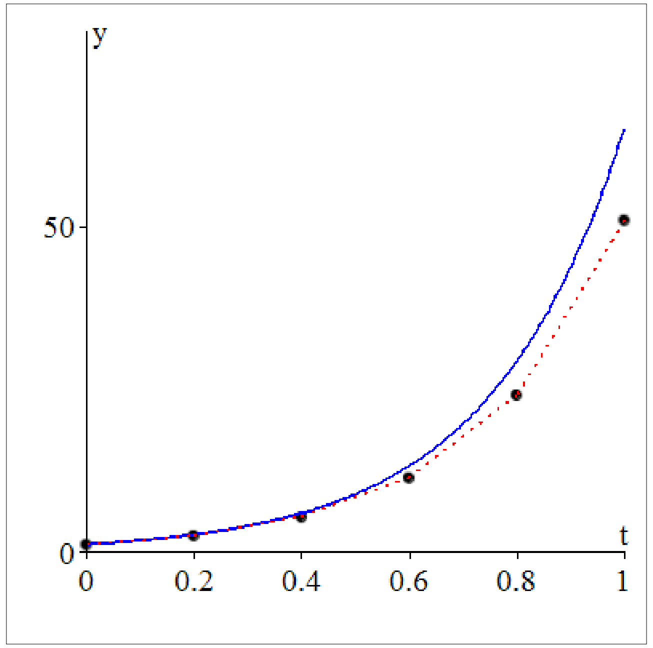
can be obtained by “integrating factor”.

Time step size: $h = 0.2$

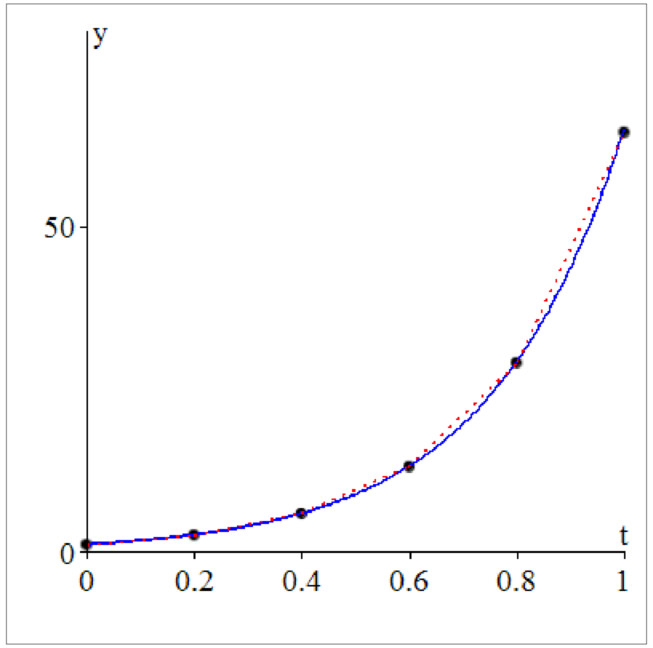
Euler



Improved Euler



Runge-Kutta



$$\frac{dy}{dt} = 1 - t + 4y$$
$$y(0) = 1$$

The exact solution

$$y(t) = -\frac{3}{16} + \frac{1}{4}t + \frac{19}{16}e^{4t}$$

can be obtained by “integrating factor”.

Time step size: $h = 0.2$

t_n	"Euler"	"Improved Euler"	"Runge-Kutta"	"Exact Solution"
0.	1.0	1.0	1.0	1.000000000
0.2	2.00	2.380000000	2.501600000	2.505329852
0.4	3.760	5.249600000	5.777635840	5.794226004
0.6	6.8880	11.27715200	12.99717789	13.05252195
0.8	12.47840	23.99956224	28.98076814	29.14487961
1.0	22.501120	50.91507195	64.44157911	64.89780316
1.2	40.5020160	107.9199526	143.1885654	144.4061208
1.4	72.86362880	228.7142995	318.1347477	321.2938588
1.6	131.0745318	484.7423149	706.8740234	714.9034825
1.8	235.8141572	1027.465708	1570.747070	1590.836532
2.0	424.3054830	2177.983301	3490.557408	3540.200110

TABLE 8.3.1

A comparison of results using the Euler and improved Euler methods for the initial value problem $y' = 1 - t + 4y$, $y(0) = 1$.

	Euler		Improved Euler		
t	$h = 0.01$	$h = 0.001$	$h = 0.025$	$h = 0.01$	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5952901	1.6076289	1.6079462	1.6088585	1.6090418
0.2	2.4644587	2.5011159	2.5020618	2.5047827	2.5053299
0.3	3.7390345	3.8207130	3.8228282	3.8289146	3.8301388
0.4	5.6137120	5.7754844	5.7796888	5.7917911	5.7942260
0.5	8.3766865	8.6770692	8.6849039	8.7074637	8.7120041
1.0	60.037126	64.382558	64.497931	64.830722	64.897803
1.5	426.40818	473.55979	474.83402	478.51588	479.25919
2.0	3029.3279	3484.1608	3496.6702	3532.8789	3540.2001

TABLE 8.3.2

A comparison of results using the improved Euler and Runge–Kutta methods for the initial value problem $\frac{dy}{dt} = 1 - t + 4y$, $y(0) = 1$.

	Improved Euler	Runge–Kutta			Exact
t	$h = 0.025$	$h = 0.2$	$h = 0.1$	$h = 0.05$	
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.6079462		1.6089333	1.6090338	1.6090418
0.2	2.5020618	2.5016000	2.5050062	2.5053060	2.5053299
0.3	3.8228282		3.8294145	3.8300854	3.8301388
0.4	5.7796888	5.7776358	5.7927853	5.7941197	5.7942260
0.5	8.6849039		8.7093175	8.7118060	8.7120041
1.0	64.497931	64.441579	64.858107	64.894875	64.897803
1.5	474.83402		478.81928	479.22674	479.25919
2.0	3496.6702	3490.5574	3535.8667	3539.8804	3540.2001