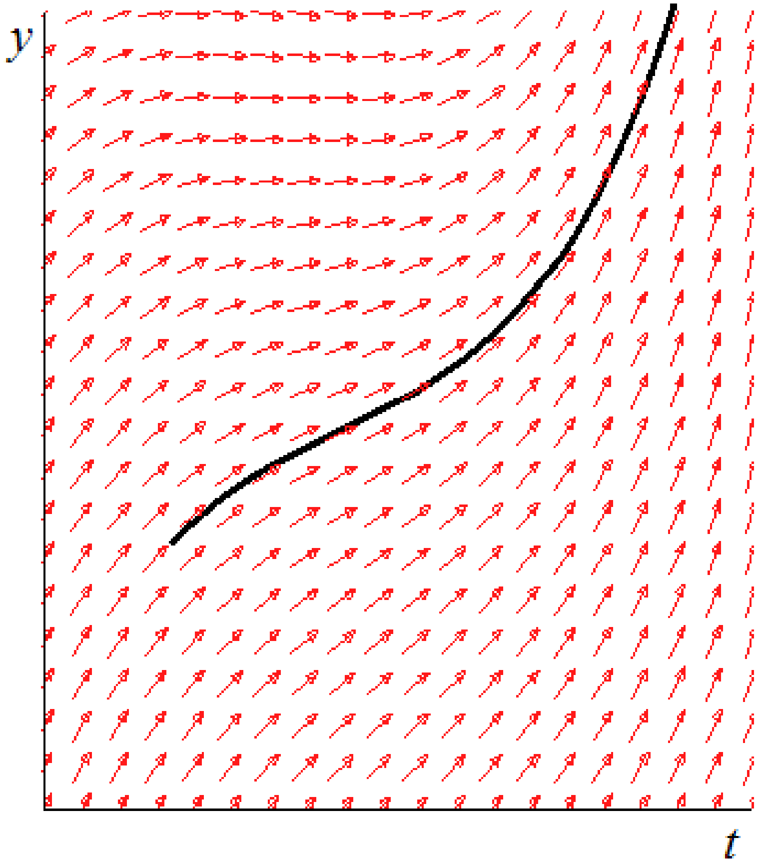
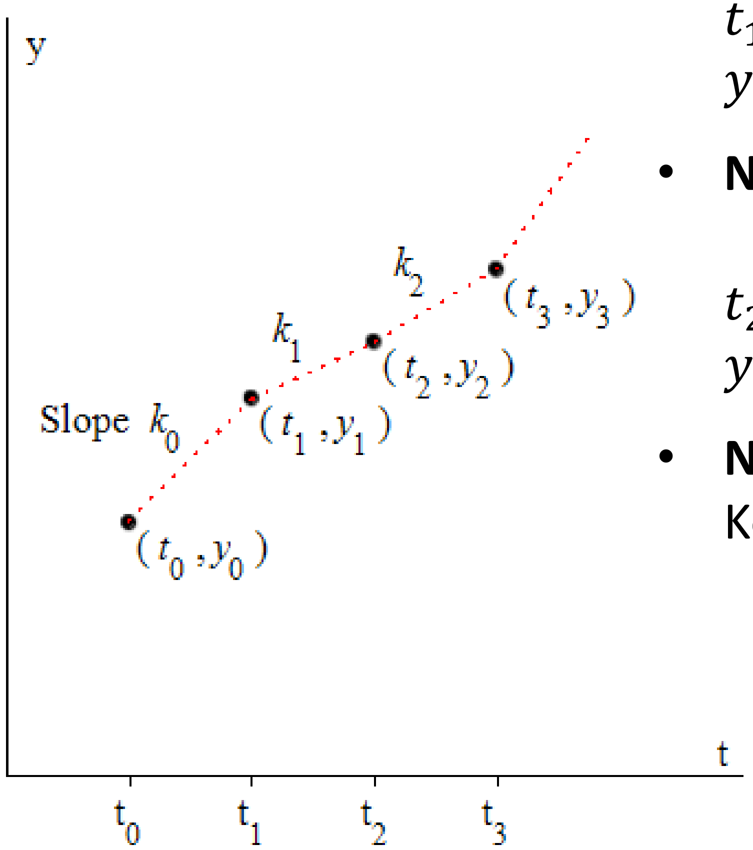
$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

Exact Solution



Euler's Method



- **Start point:** (t_0, y_0)
  $k_0 = f(t_0, y_0)$
  $t_1 = t_0 + h$
  $y_1 = y_0 + hk_0$
- **Next point:** (t_1, y_1)
  $k_1 = f(t_1, y_1)$
  $t_2 = t_1 + h$
  $y_2 = y_1 + hk_1$
- **Next point:** (t_2, y_2)
  Keep going ……..

Example:

$$\frac{dy}{dt} = 1.5 - t - 0.5y$$

$$y(0) = 0.5$$

$$f(t, y) = 1.5 - t - 0.5y$$

$h = 0.2$ (Step size)

- **Start point:** $t_0 = 0, y_0 = 0.5$

$$k_0 = f(t_0, y_0) = f(0, 0.5) = 1.5 - 0 - 0.5(0.5) = 1.25$$

$$t_1 = t_0 + h = 0.2$$
$$y_1 = y_0 + hk_0 = 0.5 + 0.2(1.25) = 0.75$$

- **Next point:** $t_1 = 0.2, y_1 = 0.75$

$$k_1 = f(t_1, y_1) = f(0.2, 0.75) = 1.5 - 0.2 - 0.5(0.75) = 0.925$$

$$t_2 = t_1 + h = 0.4$$
$$y_2 = y_1 + hk_1 = 0.75 + 0.2(0.925) = 0.935$$

- **Next point:** $t_2 = 0.4, y_2 = 0.935$

  Keep going ……..

$$\frac{dy}{dt} = 1.5 - t - 0.5y$$
$$y(0) = 0.5$$

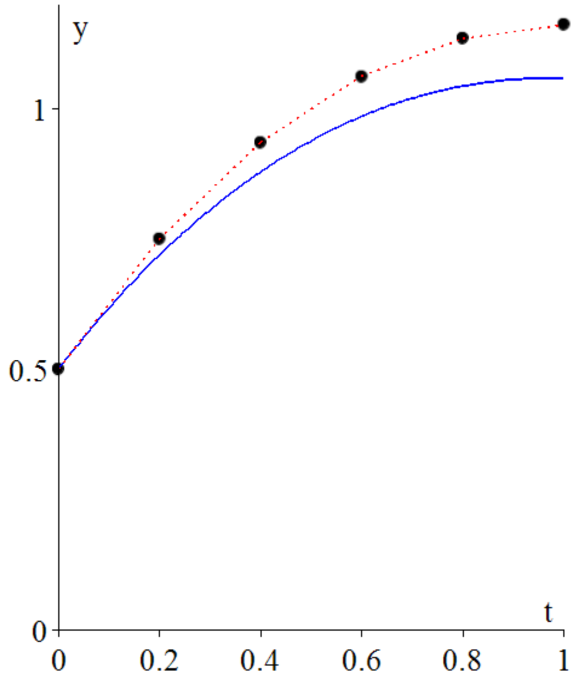
The exact solution

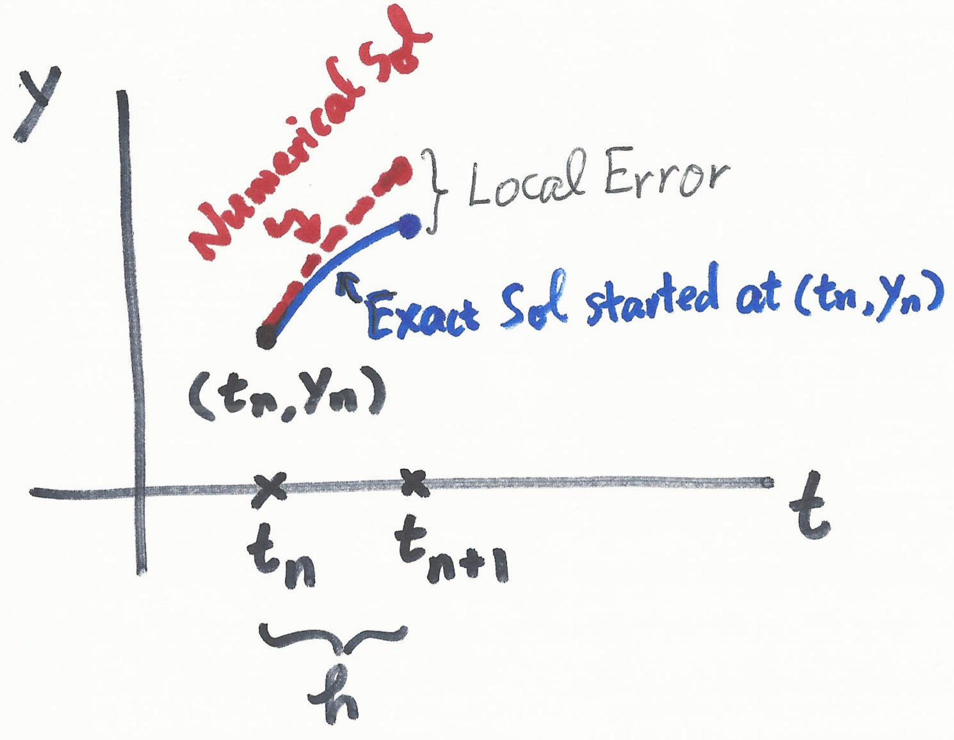
$$y(t) = 7 - 2t - 6.5e^{-0.5t}$$

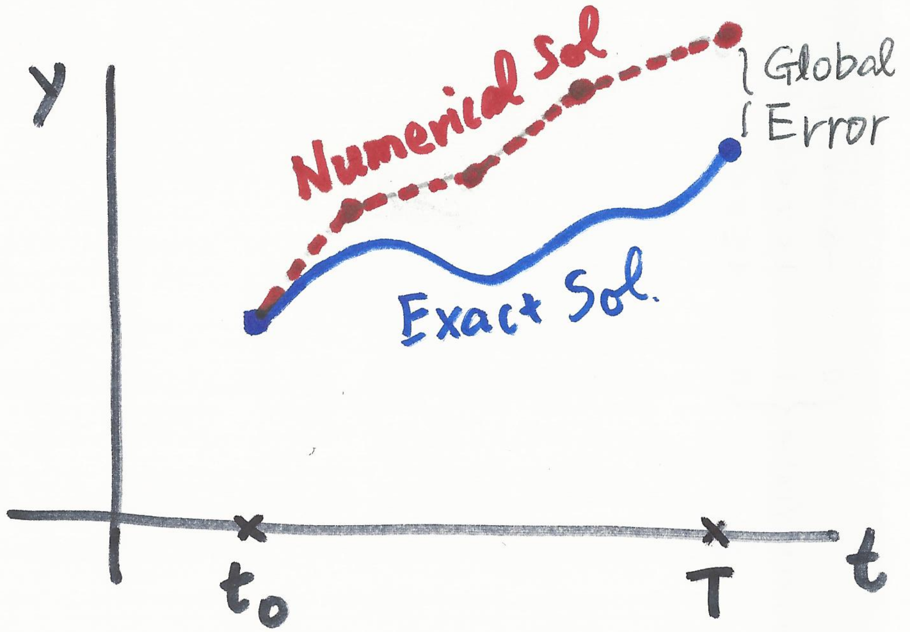
can be obtained by "integrating factor".

$h = 0.2$

t_n	Euler y_n	Exact $y(t_n)$
0	0.5	0.5
0.2	0.7500000000	0.7185567827
0.4	0.9350000000	0.8782501050
0.6	1.061500000	0.9846815656
0.8	1.135350000	1.0429197007
1.0	1.161815000	1.0575507119







Local Error

= the error generated in a single time step

Global Error

= the error accumalated over a global time interval after many steps

Local Error of Euler's Method

- From (t_n, y_n) to (t_{n+1}, y_{n+1}) :

Local Error $e_{n+1} = \frac{1}{2}\phi''(t_n^*)h^2$

of order h^2

where t_n^* is some point $t_n < t_n^* < t_{n+1}$, & $\phi(t)$: exact sol starting with (t_n, y_n)

Global Error of Euler's Method

- From (t_0, y_0) to $(t_n = T, y_n)$, after n steps:

Global Error $|E_n| \leq \frac{M}{2} \frac{e^{K(T-t_0)}-1}{K} h$

of order h

where

$$\begin{cases} K = \max\left|\frac{\partial f}{\partial y}\right| \\ M = \max\left|\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right| \end{cases}$$

$$\frac{dy}{dt} = 1.5 - t - 0.5y$$
$$y(0) = 0.5$$

The exact solution

$$y(t) = 7 - 2t - 6.5e^{-0.5t}$$

can be obtained by "integrating factor".

$h = 0.2$

t_n	Euler y_n	Exact $y(t_n)$
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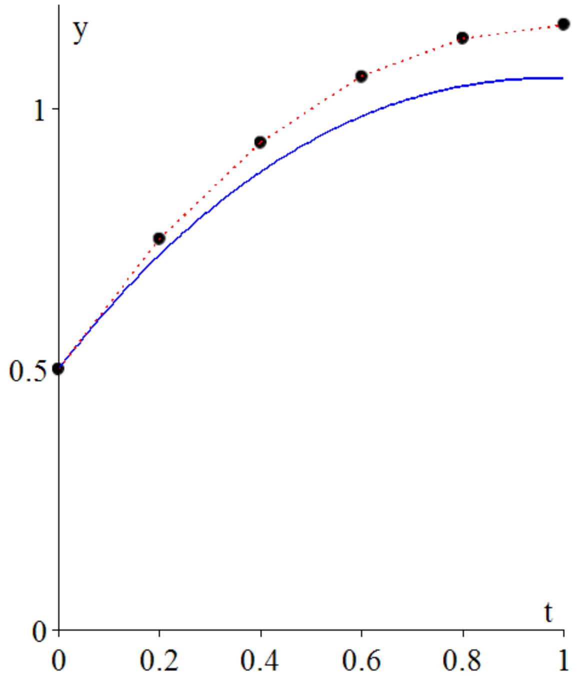
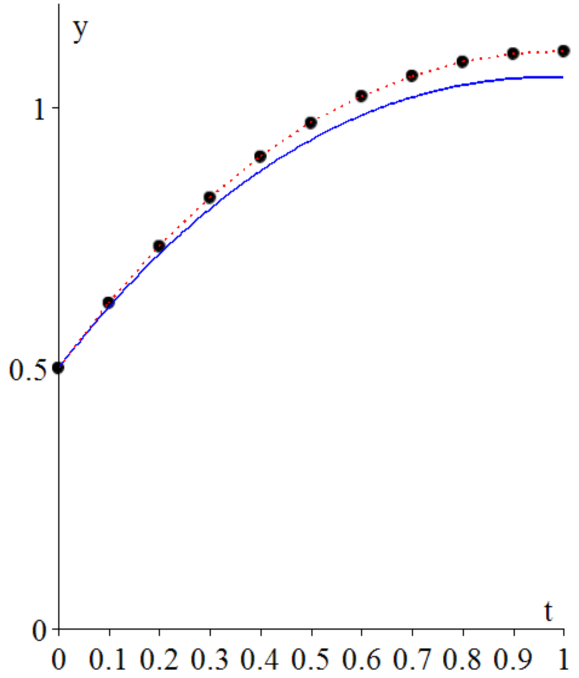
1.0	1.161815000	1.0575507119



$h = 0.1$

0	0.5	0.5
0.1	0.6250000000	0.6170087408
0.2	0.7337500000	0.7185567827
0.3	0.8270625000	0.8053981532
0.4	0.9057093750	0.8782501050
0.5	0.9704239062	0.9377949100
0.6	1.0219027109	0.9846815656
0.7	1.0608075754	1.0195274168
0.8	1.0877671966	1.0429197007
0.9	1.1033788368	1.0554170145
1.0	1.1082098950	1.0575507119



$$\frac{dy}{dt} = 1.5 - t - 0.5y$$
$$y(0) = 0.5$$

The exact solution

$$y(t) = 7 - 2t - 6.5e^{-0.5t}$$

can be obtained by "integrating factor".

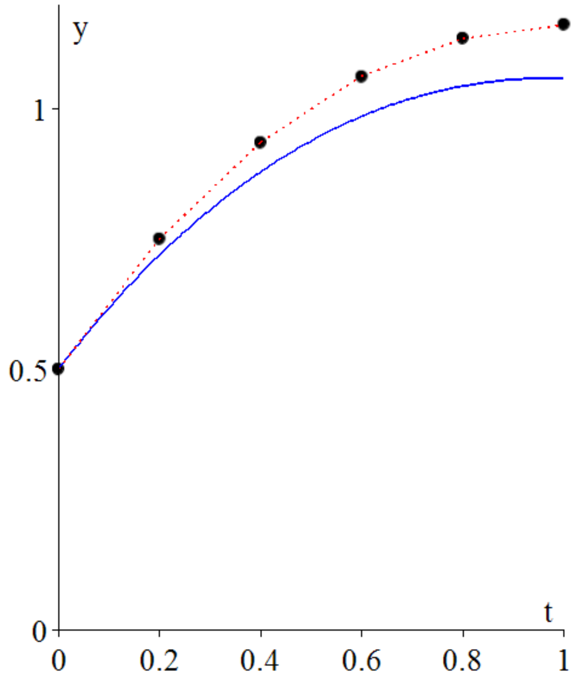
Local errors of Euler $= O(h^2)$,
Global errors of Euler $= O(h)$.

From $h = 0.2$ to $h = 0.1$:

- The local errors of Euler are about 4 times smaller;
- the global errors of Euler are about 2 times smaller.

$h = 0.2$

t_n	Euler y_n	Exact $y(t_n)$
0	0.5	0.5
0.2	0.7500000000	0.7185567827
0.4	0.9350000000	0.8782501050
0.6	1.061500000	0.9846815656
0.8	1.135350000	1.0429197007
1.0	1.161815000	1.0575507119



$h = 0.1$

0	0.5	0.5
0.1	0.6250000000	0.6170087408
0.2	0.7337500000	0.7185567827
0.3	0.8270625000	0.8053981532
0.4	0.9057093750	0.8782501050
0.5	0.9704239062	0.9377949100
0.6	1.0219027109	0.9846815656
0.7	1.0608075754	1.0195274168
0.8	1.0877671966	1.0429197007
0.9	1.1033788368	1.0554170145
1.0	1.1082098950	1.0575507119