The Linear Approximating System of a Nonlinear System Near an Equilibrium

[1] Consider

$$\begin{cases} x_1' = x_1 - x_1(x_1^2 + x_2^2 + x_3^2), \\ x_2' = -x_2 - x_2(x_1^2 + x_2^2 + x_3^2), \\ x_3' = -2x_3 - x_3(x_1^2 + x_2^2 + x_3^2). \end{cases} \tag{$*$}$$

(a) Find all equilibrium solutions of the system $(*)$.

(b) For each equilibrium point,

- give the linear approximating system near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(*)$.

[2] Consider

$$\begin{cases} x_1' = -x_1 + x_1(x_1^2 + x_2^2 + x_3^2), \\ x_2' = x_2 + x_2(x_1^2 + x_2^2 + x_3^2), \\ x_3' = 2x_3 + x_3(x_1^2 + x_2^2 + x_3^2). \end{cases} \tag{$*$}$$

(Notice that the right sides of this system are the same as those in [1], except that the signs are switched.)

(a) Find all equilibrium solutions of the system $(*)$.

(b) For each equilibrium point,

- give the linear approximating system near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(*)$.

[3] Consider

$$\begin{cases} x_1' = -2x_1 - x_2 - x_1x_3, \\ x_2' = x_1 - 2x_2 + x_2x_3, \\ x_3' = -x_3 - x_1^2 - x_2^2 - x_3^2. \end{cases} \tag{$*$}$$

(a) Find all equilibrium solutions of the system $(*)$.

(b) For each equilibrium point,

- give the linear approximating system near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(*)$.

Turn over for the answers

Answers:

[1] (a) $(0,0,0),(1,0,0),(-1,0,0)$.

(b) Near $(0,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(0,0,0)$ is unstable.

Near $(1,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(1,0,0)$ is asymptotically stable.

Near $(-1,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 + 1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(-1,0,0)$ is asymptotically stable.

[2] (a) $(0,0,0),(1,0,0),(-1,0,0)$.

(b) Near $(0,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(0,0,0)$ is unstable.

Near $(1,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(1,0,0)$ is unstable.

Near $(-1,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 + 1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(-1,0,0)$ is unstable.

[3] (a) $(0,0,0),(0,0,-1)$.

(b) Near $(0,0,0)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(0,0,0)$ is asymptotically stable.

Near $(0,0,-1)$: the linear approximating system is $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 + 1 \end{bmatrix}$.

$(0,0,-1)$ is unstable.