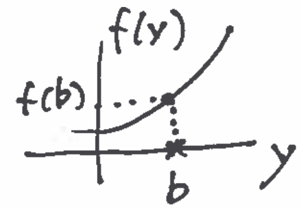
Linear Approximating Systems Near Equilibria

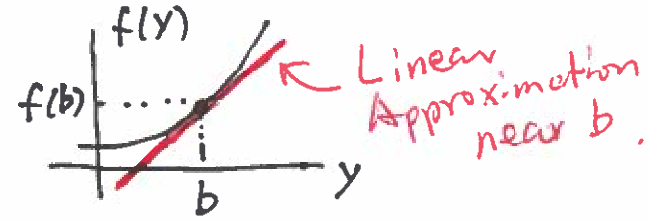
Review of Calculus



Given a nonlinear function $f(y)$,

the Linear Approximation of f near $y \approx b$:

$$f(y) \approx f(b) + \left[\frac{df}{dy}(b)\right](y-b) \quad \text{for } y \approx b.$$

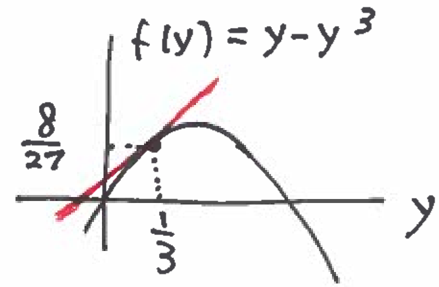


Example $f(y) = y - y^3$ has derivative $\frac{df}{dy}(y) = 1 - 3y^2$.

- At $y = \frac{1}{3}$: $f(\frac{1}{3}) = \frac{8}{27}$, $\frac{df}{dy}(\frac{1}{3}) = \frac{2}{3}$

  $\Rightarrow$ Lin. Approximation of f near $y \approx \frac{1}{3}$:

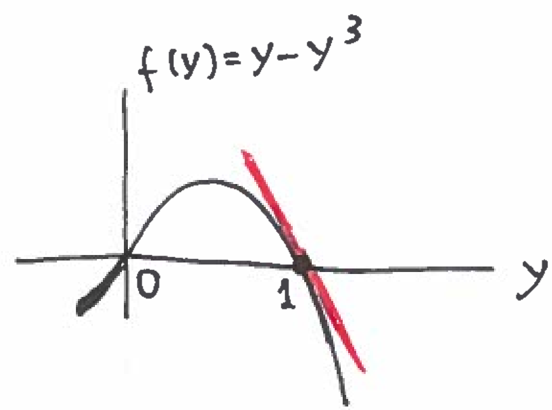
  $$f(y) \approx \frac{8}{27} + \frac{2}{3}\left(y - \frac{1}{3}\right) \quad \text{for } y \approx \frac{1}{3}$$



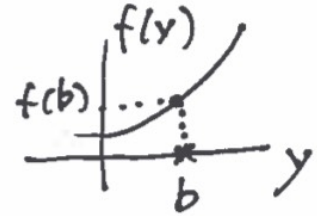
- At $y = 1$: $f(1) = 0$, $\frac{df}{dy}(1) = -2$

  $\Rightarrow$ Lin. Approximation of f near $y \approx 1$:

  $$f(y) \approx -2(y-1) \quad \text{for } y \approx 1.$$



Review of Calculus (Single Variable)



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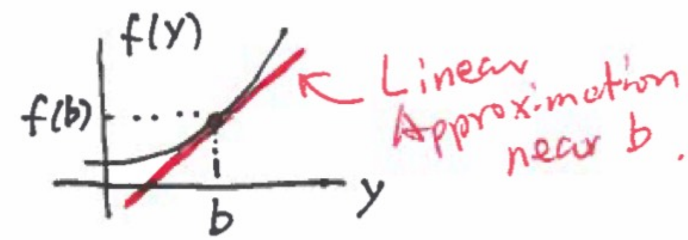

Review of Calculus (Multivariables)

Given a function $f(x_1, x_2, \cdots\cdots, x_n)$,

the **linear approximating function** of f near $(x_1, \cdots, x_n) \approx (a_1, \cdots, a_n)$ is given by:

$$f(x_1, x_2, \cdots, x_n) \approx f(a_1, \cdots, a_n) + \frac{\partial f}{\partial x_1}(a_1, \cdots, a_n)(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(a_1, \cdots, a_n)(x_n - a_n).$$

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<u>Example</u>: $f(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^3$ near $(x_1, x_2) \approx (-1, 2)$.

The Linear Approximation of f Near $(x_1, x_2) \approx (-1, 2)$:

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$$f(-1, 2) = (-1)^2 - 4(-1)(2) + 2^3 = 17.$$

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, x_2) = 2x_1 - 4x_2 \\ \frac{\partial f}{\partial x_2}(x_1, x_2) = -4x_1 + 3x_2^2 \end{cases} \qquad \begin{cases} \frac{\partial f}{\partial x_1}(-1, 2) = 2(-1) - 4(2) = -10 \\ \frac{\partial f}{\partial x_2}(-1, 2) = -4(-1) + 3(2)^2 = 16 \end{cases}$$

The <u>Linear Approximation</u> of f Near $(x_1, x_2) \approx (-1, 2)$:

$$f(x_1, x_2) \approx 17 - 10(x_1 + 1) + 16(x_2 - 2)$$

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<u>Example</u>. $f(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^3$ near $(x_1, x_2) \approx (9, 3)$.

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<u>Example</u>. $f(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^3$ near $(x_1, x_2) \approx (9, 3)$.

$$f(9,3) = (9)^2 - 4(9)(3) + 3^3 = \mathbf{0}.$$

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, x_2) = 2x_1 - 4x_2 \\ \frac{\partial f}{\partial x_2}(x_1, x_2) = -4x_1 + 3x_2^2 \end{cases} \qquad \begin{cases} \frac{\partial f}{\partial x_1}(9,3) = 2(9) - 4(3) = \mathbf{6} \\ \frac{\partial f}{\partial x_2}(9,3) = -4(9) + 3(3)^2 = \mathbf{-9} \end{cases}$$

The <u>Linear Approximation</u> of f Near $(x_1, x_2) \approx (9, 3)$:

$$f(x_1, x_2) \approx \mathbf{6}(x_1 - 9) - \mathbf{9}(x_2 - 3)$$

Consider an n-D system of autonomous diff eqs:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, \cdots, x_n) \\ \dfrac{dx_2}{dt} = f_2(x_1, \cdots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, \cdots, x_n) \end{cases}$$

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$$

• <u>Definition</u> An equilibrium is, by definition, a constant sol. (i.e., a time-independent sol.)

$$\begin{cases} f_1(x_1, \cdots, x_n) = 0 \\ f_2(x_1, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, \cdots, x_n) = 0 \end{cases}$$

• <u>Question</u> Let $(x_1, \cdots, x_n) = (a_1, a_2, \cdots, a_n)$ be an equilibrium. Is it stable, asymptotically stable, or unstable?

Jacobian Matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Linear Approximating System Near an equilibrium $(a_1, \cdots, a_n)$

$$\frac{d\vec{x}}{dt} = \boxed{}\,(\vec{x} - \vec{a})$$

↑ an $n \times n$ constant matrix

J evaluated at $(a_1, \cdots, a_n)$

Recall basics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Eigenvalues & (generalized) eigenvectors of A
$\Rightarrow$ solution formulas, dynamics, stability/instability,....

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$$\left.\begin{array}{l}\text{Negative eigenvalues } \lambda < 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda < 0\end{array}\right\}\text{help stabilization.}$$

$$\left.\begin{array}{l}\text{Zero eigenvalues } \lambda = 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda = 0\end{array}\right\}\text{are “neutral”.}$$

$$\left.\begin{array}{l}\text{Positive eigenvalues } \lambda > 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda > 0\end{array}\right\}\text{imply instability.}$$

① • If all eigenvalues of the lin. approx. system have $\text{Re}\,\lambda < 0$,
then the equilibrium $(a_1, a_2, \cdots, a_n)$ is asympt. stable with respect to the nonlin. system.

② • If one of the eigenvalues of the lin. approx. system have $\text{Re}\,\lambda > 0$,
then the equilibrium $(a_1, \cdots, a_n)$ is unstable with respect to the nonlin. system.

③ • If all eigenvalues of the lin. approx. system have $\text{Re}\,\lambda \leq 0$
& some eigenvalues have $\text{Re}\,\lambda = 0$,
then the lin. approx. method is inconclusive in determining the stability/instability with respect to the nonlin. system.
(Other more advanced methods are needed.)

④ · If all eigenvalues of the lin. approx. system are NOT neutral (i.e. $Re\,\lambda \neq 0$ for all eigenvalues), the local nonlinear dynamics near the equilibrium is "essentially equivalent" to the linear approximating dynamics.

⑤ If one of the eigenvalues of the lin. approx. system is neutral (i.e. $Re\,\lambda = 0$ for at least one eigenvalue), the local nonlinear dynamics near the equilibrium may be non-equivalent to the lin. approx. dynamics.

Example 5.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find all equilibria.
- For each equilibrium, give the linear approximating system near it.
- Sketch the phase portrait of the linear approximating system.
- Determine whether each equilibrium is stable or unstable with respect to the nonlinear system.

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- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

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$$\begin{cases} (1) \quad -x_1 - x_2 = 0 \\ (2) \quad x_1 - 7x_2 + x_2^2 - 3x_1x_2 = 0 \end{cases}$$

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From (1), $x_2 = -x_1$.
Substitute this in (2): $8x_1 + 4x_1^2 = 0 \Rightarrow x_1 = 0$, or $x_1 = -2$.

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Combined with $x_2 = -x_1$:

$\Rightarrow$ Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.

Example 5. Linear approximating system near the equilibrium $(0,0)$.

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  - Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $J = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix}$
  - The linear approximating system near $(0, 0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 5. Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

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- Linear approx system near $(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors: $$\lambda_1 = -4 + 2\sqrt{2} < 0,\ \ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix},$$ $$\lambda_2 = -4 - 2\sqrt{2} < 0,\ \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix}$$

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$$\lambda_1 = -4 + 2\sqrt{2} < 0,\ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix},$$

$$\lambda_2 = -4 - 2\sqrt{2} < 0,\ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix}$$

- Thus, $(0,0)$ is an attractive node & is asymptotically stable in the linear dynamics.

Example 5. Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = -x_1 - x_2 \\ f_2(x_1,x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1,x_2) = (0,0)$, $(x_1,x_2) = (-2,2)$.

- Linear approx system near $(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues & eigenvectors:
$$\lambda_1 = -4 + 2\sqrt{2} < 0,\ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3+2\sqrt{2} \\ 1 \end{bmatrix},$$
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- Since the eigenvalues are not neutral,
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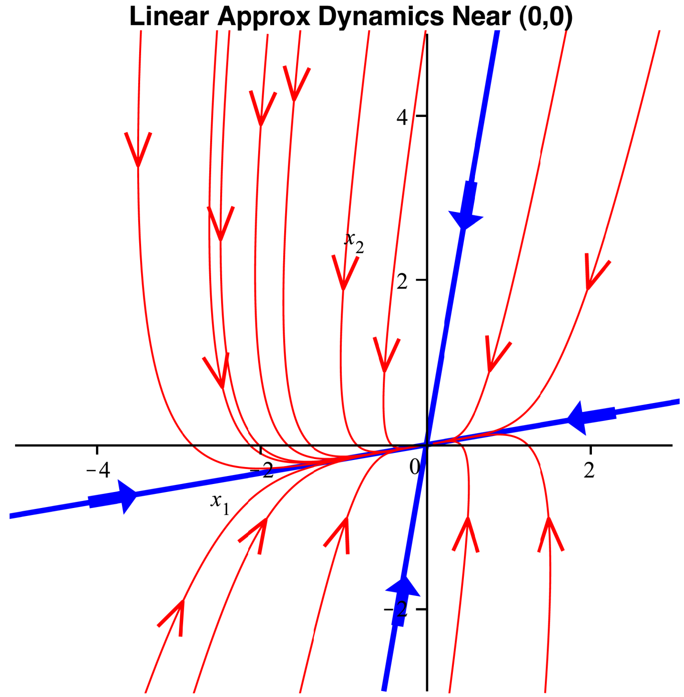
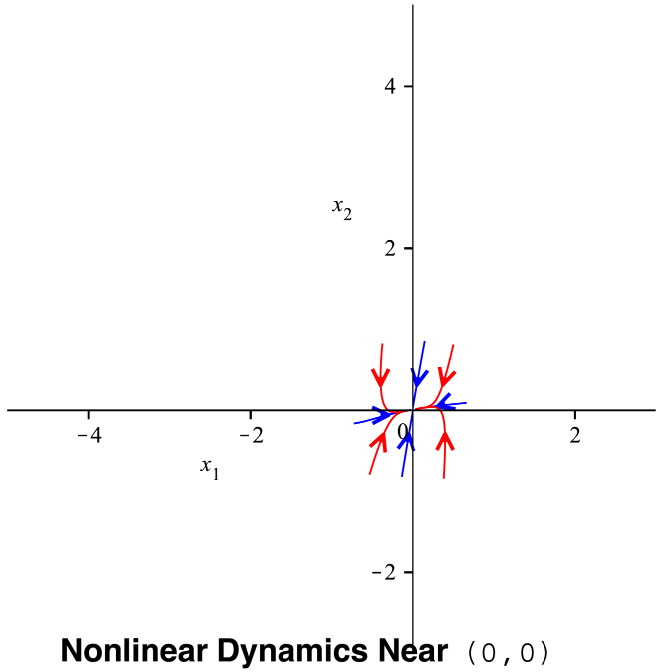
Example 5. Since all the eigenvalues are non-neutral,
Linear approx dynamics ⇒ Nonlinear local dynamics near equilibria





Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

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the nonlinear dynamics are equivalent to the linear dynamics near $(-2, 2)$.

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- Thus, $(-2, 2)$ is a saddle & is unstable in the linear dynamics.

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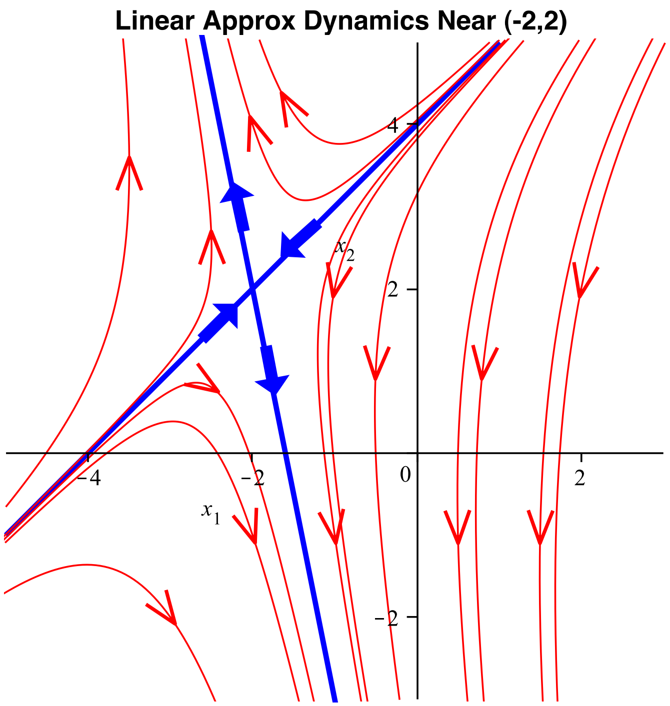
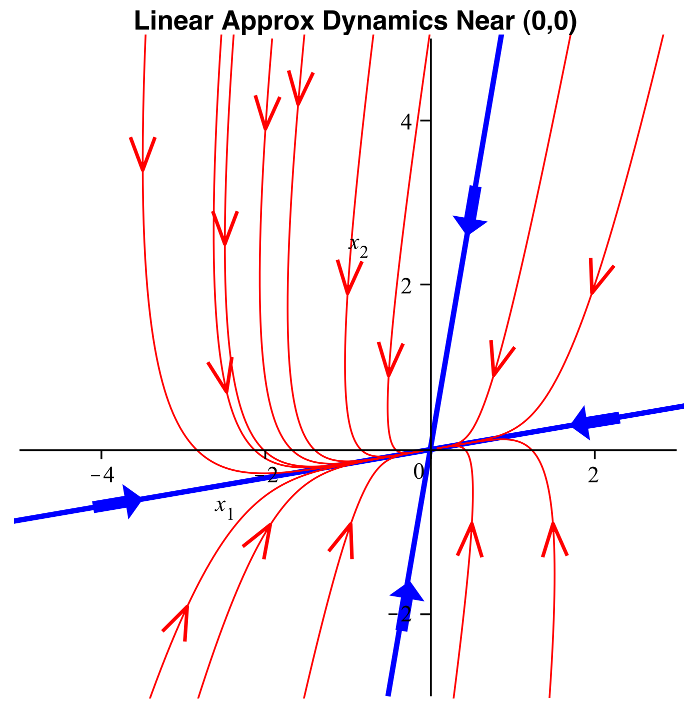
the nonlinear dynamics are equivalent to the linear dynamics near $(-2, 2)$.

- Equilibrium $(-2, 2)$ is also a saddle with respect to the original nonlinear system & it is unstable.

Example 5. Since all the eigenvalues are non-neutral,

Linear approx dynamics $\Rightarrow$ Nonlinear local dynamics near equilibria





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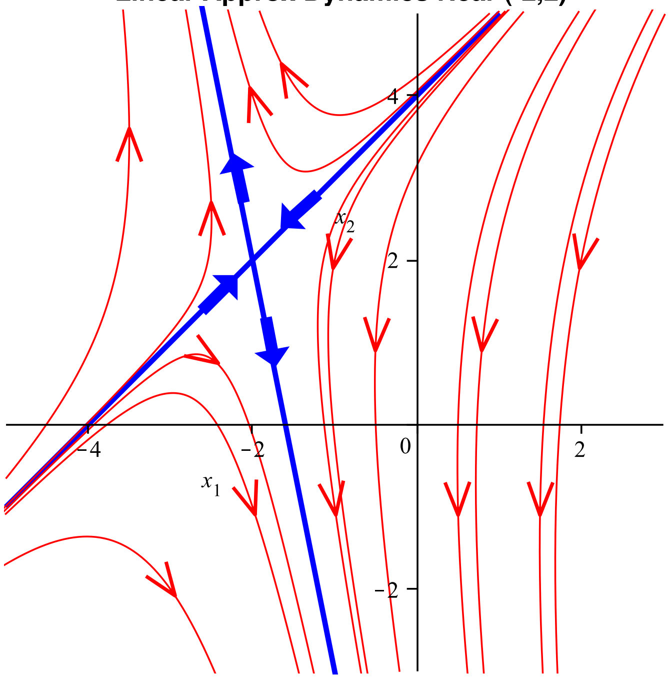
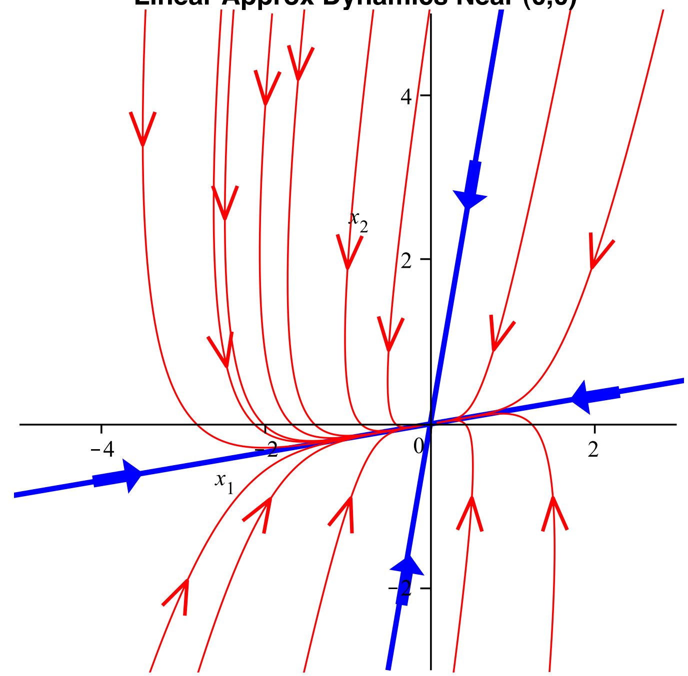
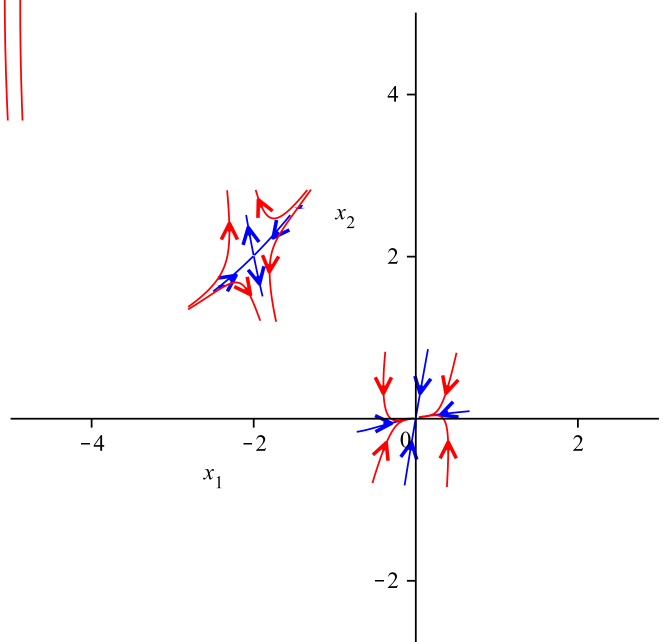
Linear Approx Dynamics Near (-2,2)



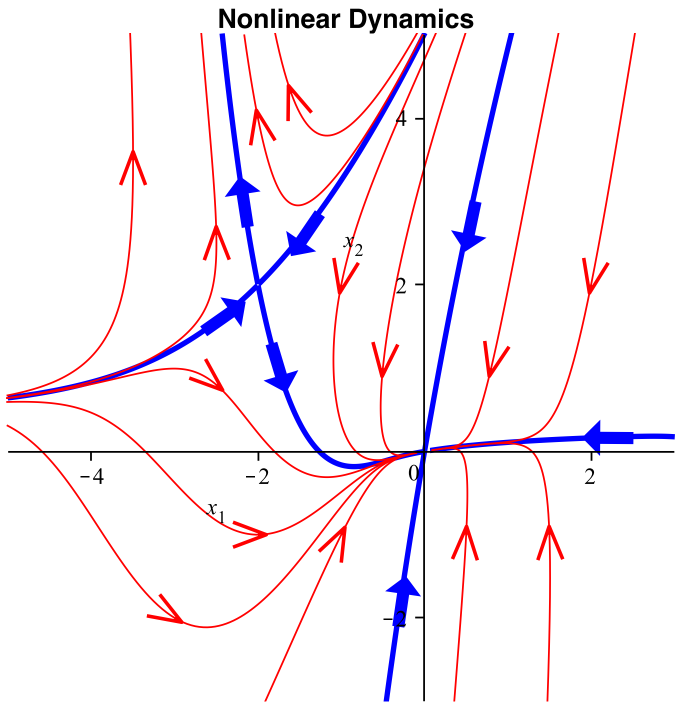
Linear Approx Dynamics Near (0,0)





Nonlinear Dynamics Near Equilibria

Example 5. Global phase portrait of the nonlinear system



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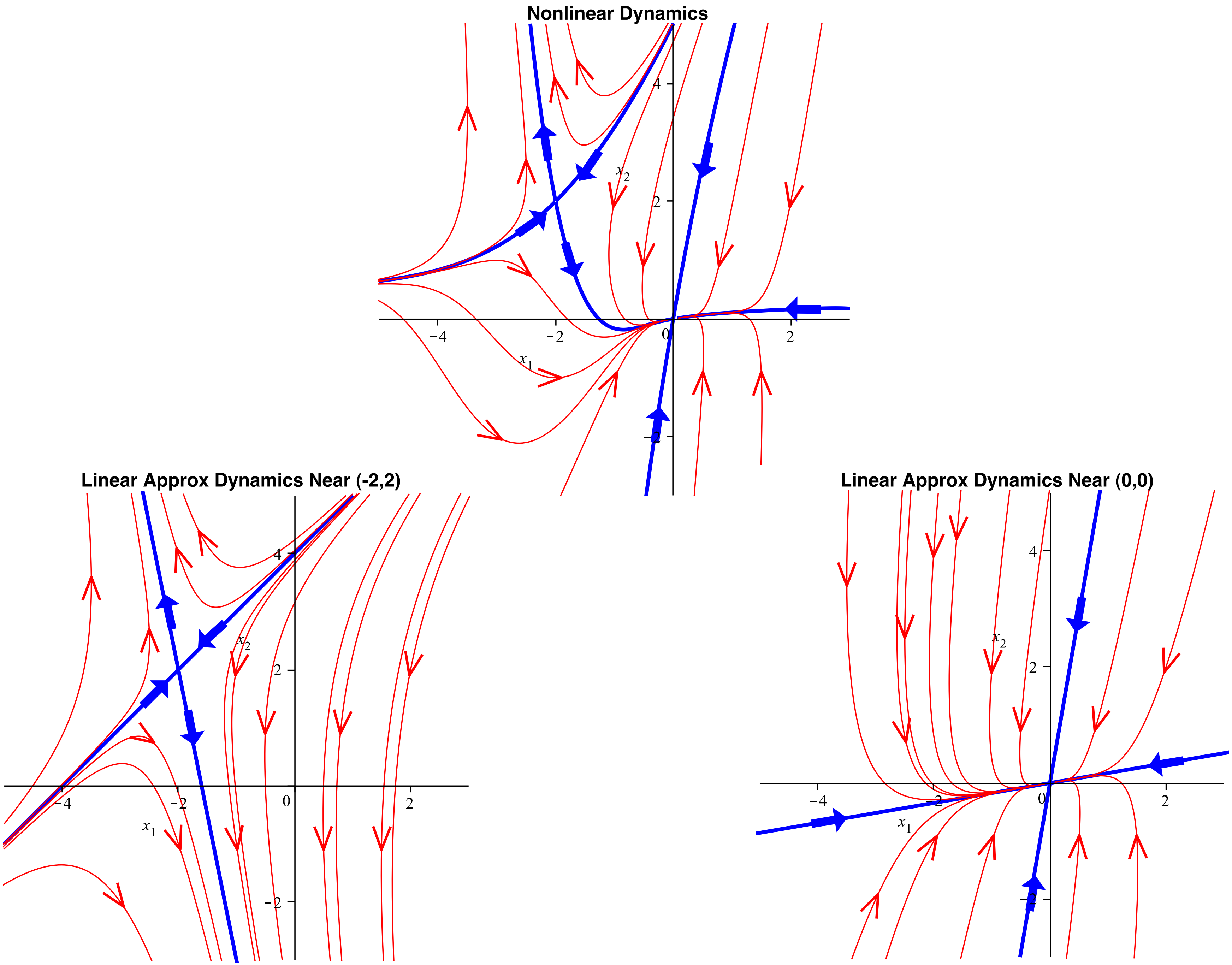

Example 6 (Neutral Eigenvalue)

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find all equilibria.
- For each equilibrium, give the linear approximating system near it.
- Sketch the phase portrait of the linear approximating system.
- Determine whether each equilibrium is stable or unstable with respect to the nonlinear system.

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

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$$\begin{cases} (1) \quad 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 = 0 \\ (2) \quad -x_2 + x_1^2 = 0 \end{cases}$$

Example 6 (continued). Find equilibria.

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From (2), $x_2 = x_1^2$.

Substitute this in (1):

$$x_1^3 + x_1^4 + x_1^5 = 0 \Rightarrow x_1^3(1 + x_1 + x_1^2) = 0 \Rightarrow x_1 = 0$$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

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From $x_2 = x_1^2$ it follows $x_2 = 0$.

$\Rightarrow$ **Only one equilibrium:** $(x_1, x_2) = (0, 0)$.

Example 6 (continued).

Linear approx sys near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

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Linear approx sys near the equilibrium $(0,0)$.

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- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_1^2 + 5x_1^4 & 2x_1 + 2x_2 \\ 2x_1 & -1 \end{bmatrix}$$

Example 6 (continued).

Linear approx sys near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

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- Evaluate J at equilibrium $(x_1, x_2) = (0,0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

Example 6 (continued).

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- Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$
- The linear approximating system near $(0, 0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 6. (continued)

Linear Approximate Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0, 0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$

Example 6. (continued)

Linear Approximate Dynamics near $(0, 0)$.

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- Linear approx system near the equilibrium$(0, 0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:

$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

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Linear Approximate Dynamics near $(0,0)$.

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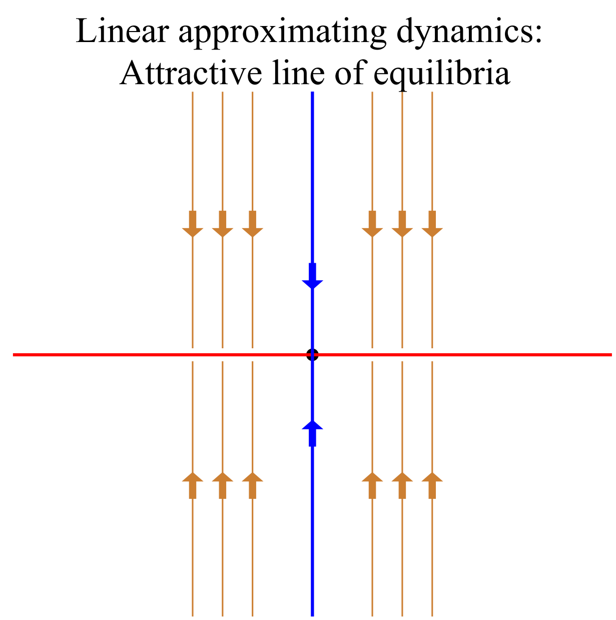
- Thus, the linear approximate dynamics has an attractive line of equilibria.



Example 6. (continued)

Linear Approximate Dynamics near $(0,0)$.

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- Linear approx system near the equilibrium$(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:

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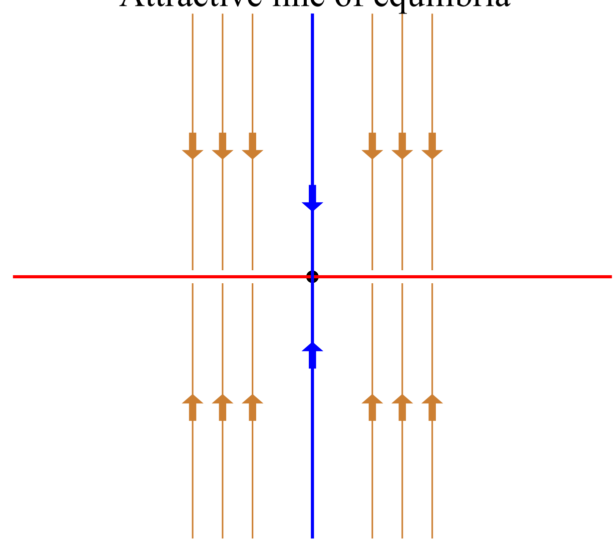
- Thus, the linear approximate dynamics has an attractive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0,0)$.

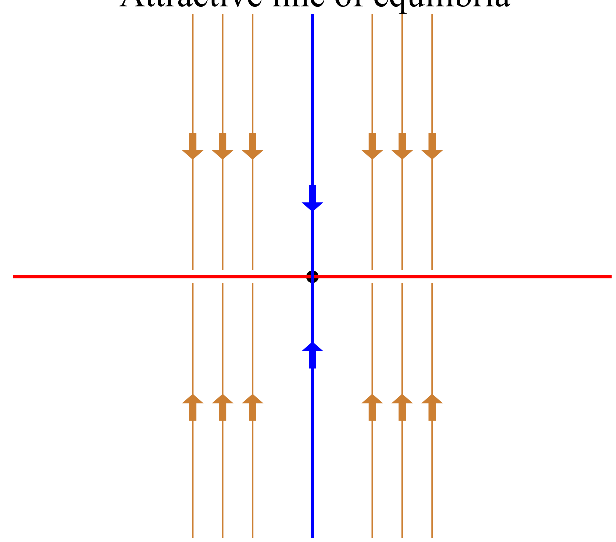


Example 6. (continued)

Linear Approximate Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0, 0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
- Thus, the linear approximate dynamics has an attractive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0, 0)$.
- In other words, the linear analysis fails to determine the local nonlinear dynamics near $(0, 0)$.

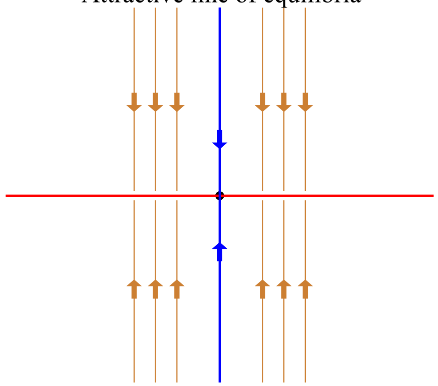


Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$$



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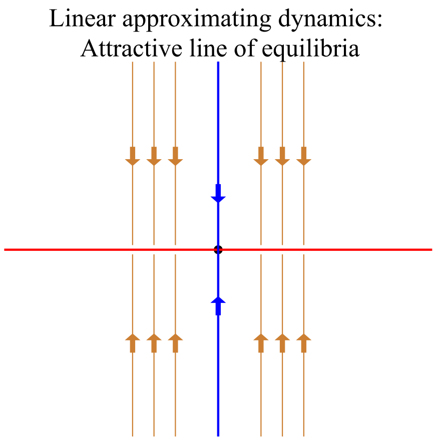
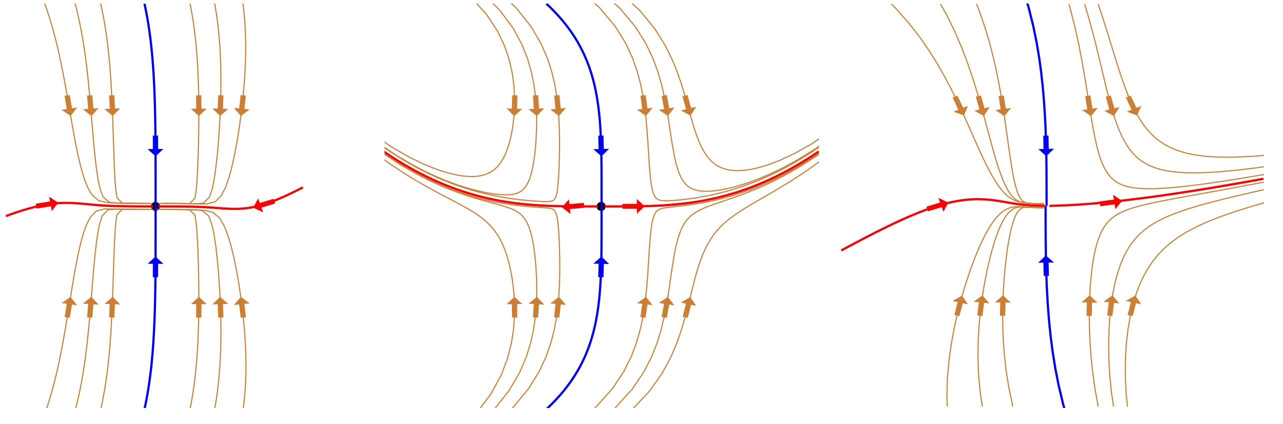


The following is an *incomplete* list of the possible local phase portraits of the nonlinear system near $(0,0)$:



et cetera

Example 6. (continued)

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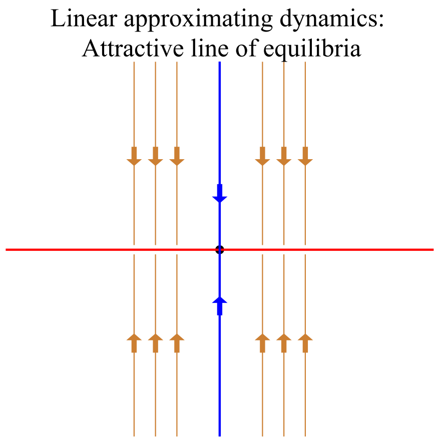

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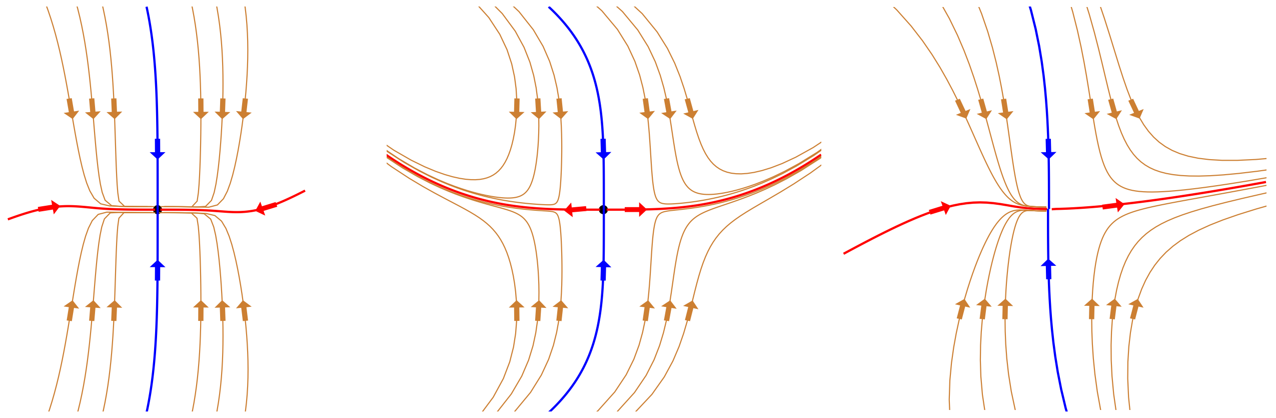


et cetera

To determine the correct picture, need advanced nonlinear theories: normal forms, center manifolds, $\cdots$.

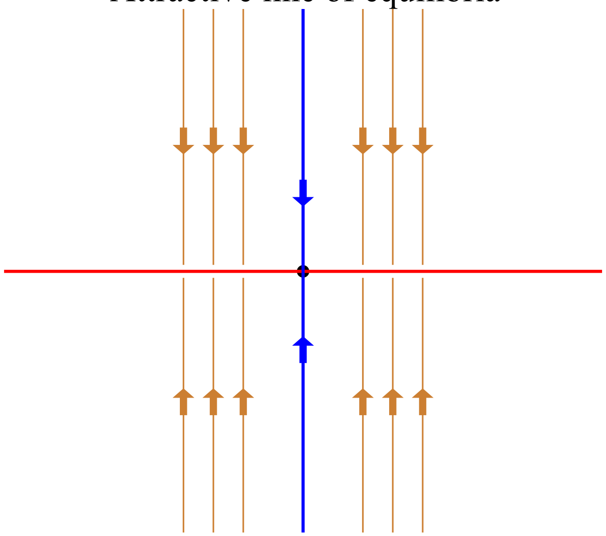
Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$$

Linear approximating dynamics:
Attractive line of equilibria



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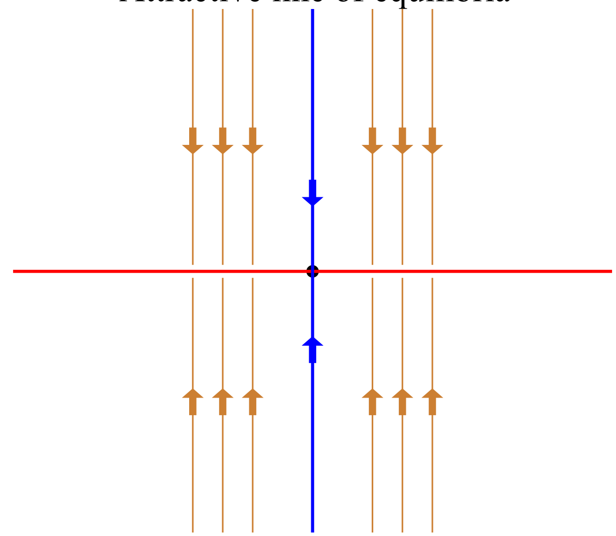
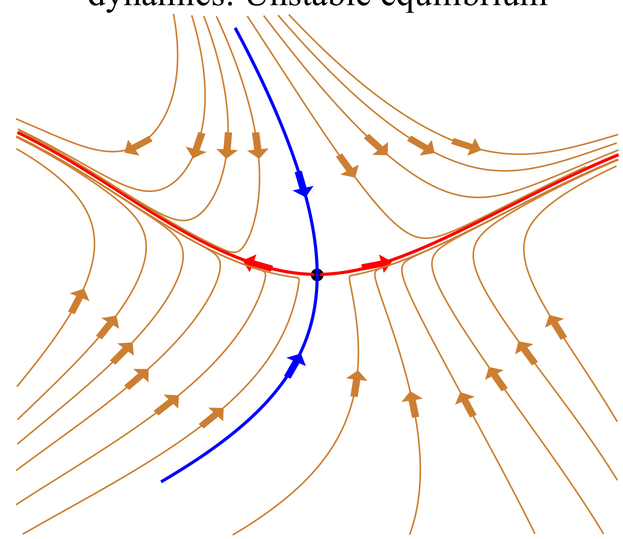


The actual local phase portrait of the nonlinear system near $(0,0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix}$$



Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0,0)$:

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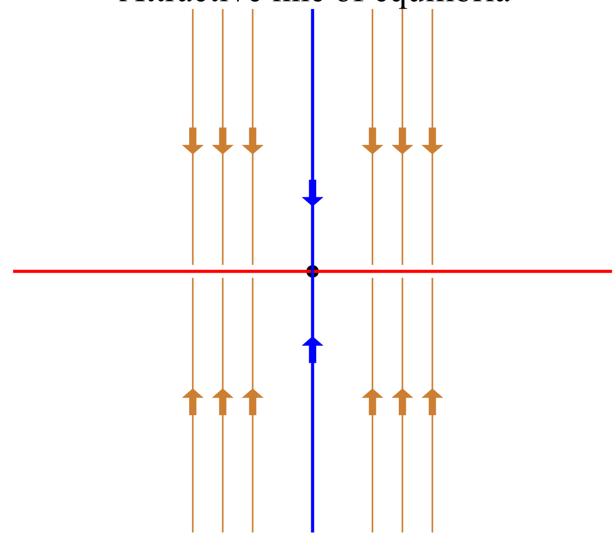
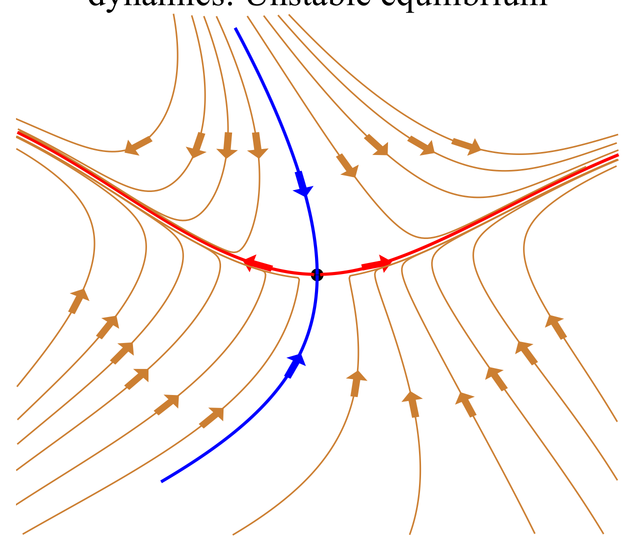
Linear approximating dynamics:
Attractive line of equilibria



Local phase portrait of nonlinear
dynamics: Unstable equilibrium



The actual local phase portrait of the nonlinear system near $(0,0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix}$$

- Impossible to get this by the linear approximation alone.
- Advanced *nonlinear* tools (center manifolds, ...) can get us this picture.

Example 7 (Neutral Eigenvalue)

$$\begin{cases} x_1' = x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2' = x_2 + \frac{1}{2} x_1^3 \end{cases}$$

- Give the linear approximating system near the equilibrium $(0, 0)$. Sketch the phase portrait of the linear approx system.
- Determine whether $(0, 0)$ is stable or unstable with respect to the nonlinear system.
- Sketch the local phase portrait of the nonlinear system near $(0, 0)$

Example 7 (a) Linear approx system near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

Example 7 (a) Linear approx system near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + 4x_1^3 & x_1 + \frac{3}{5}x_2^2 \\ \frac{3}{2}x_1^2 & 1 \end{bmatrix}$$

Example 7 (a) Linear approx system near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

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- Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

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- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + 4x_1^3 & x_1 + \frac{3}{5}x_2^2 \\ \frac{3}{2}x_1^2 & 1 \end{bmatrix}$$

- Evaluate J at equilibrium $(x_1, x_2) = (0,0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- The linear approximating system near $(0,0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 7 (a) Linear Approx Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

- Linear approx system near the equilibrium$(0, 0)$:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = 1 > 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

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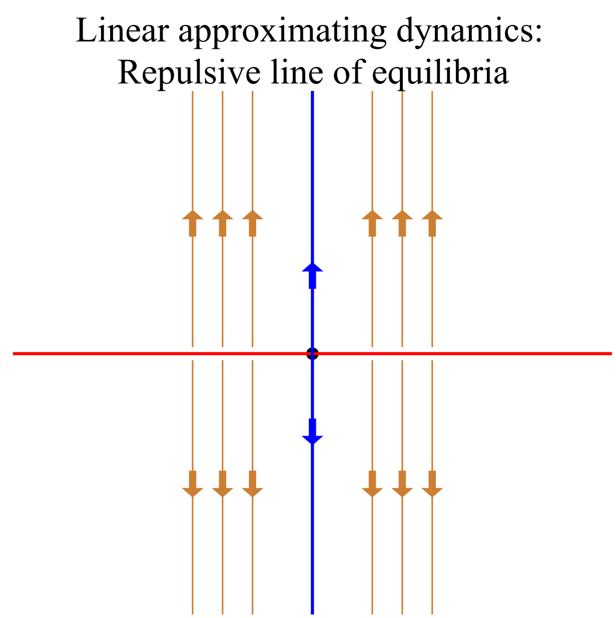
- Thus, the linear approximate dynamics has a repulsive line of equilibria.



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- Eigenvalues & eigenvectors:
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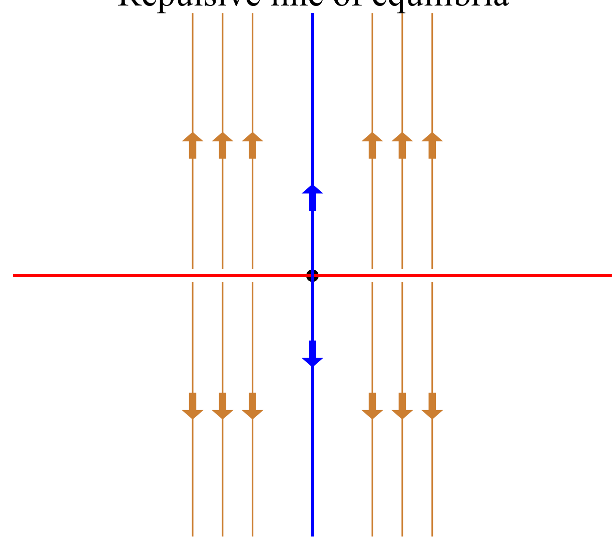
- Thus, the linear approximate dynamics has a repulsive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0,0)$.

Linear approximating dynamics:
Repulsive line of equilibria

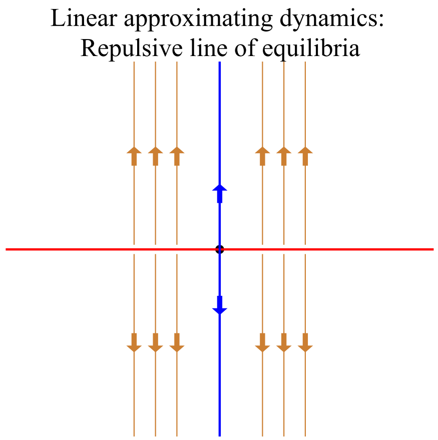


Example 7. (b)(c) Local nonlinear dynamics near $(0, 0)$

Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$



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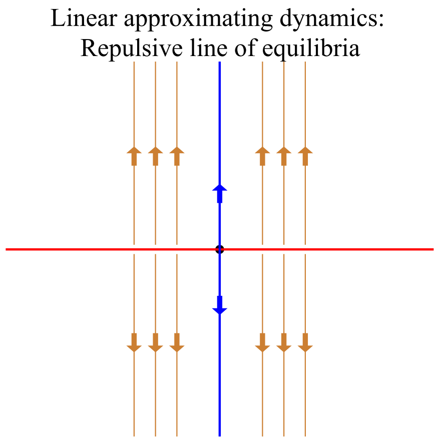
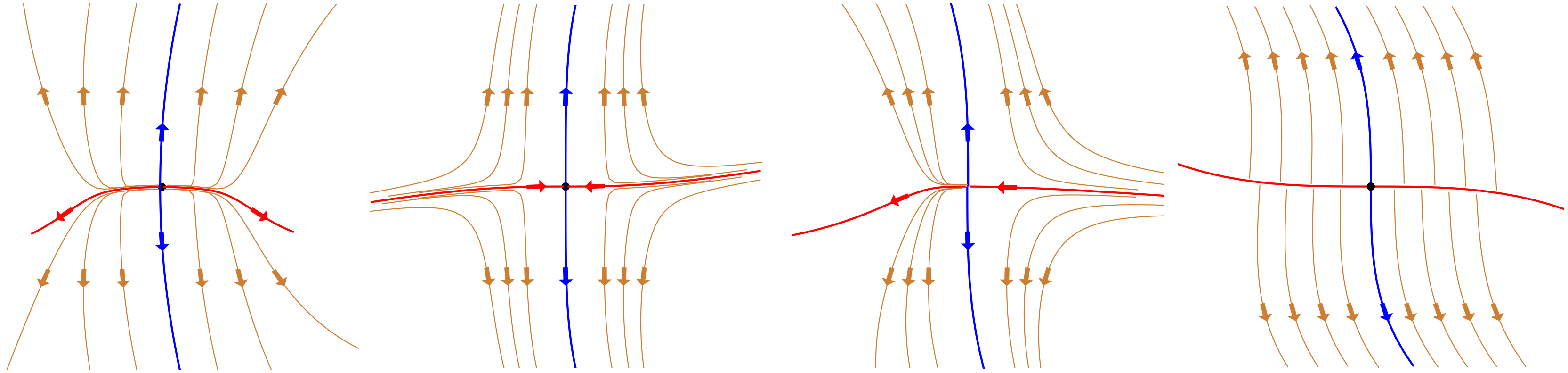


An *incomplete* list of possible nonlinear dynamics near $(0, 0)$:



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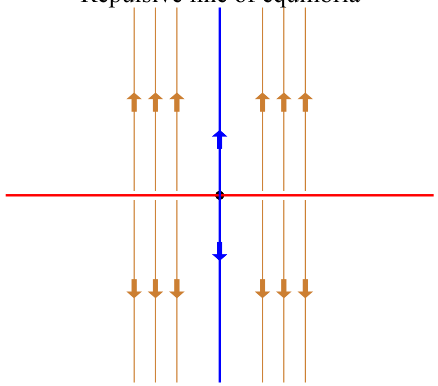

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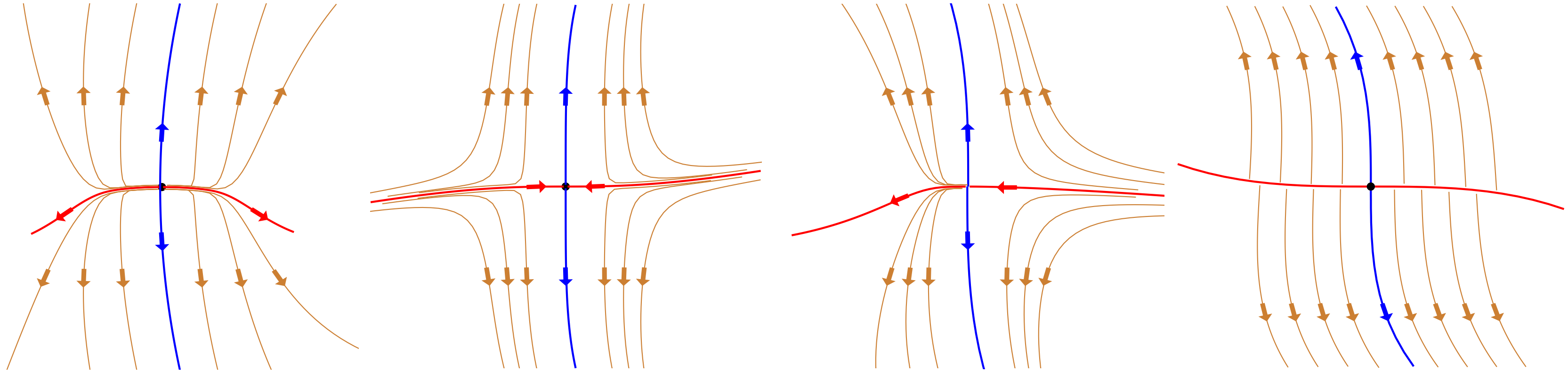

- Linear analysis alone cannot determine the correct picture.

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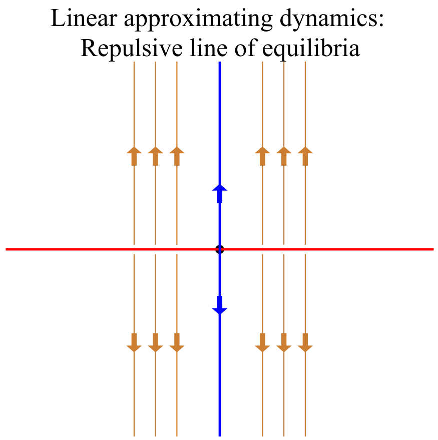

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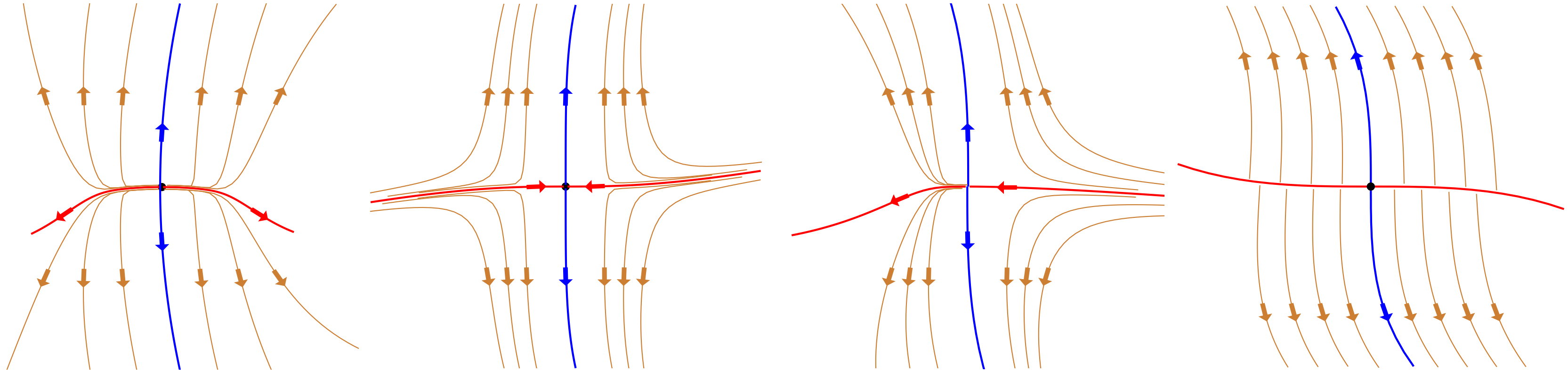
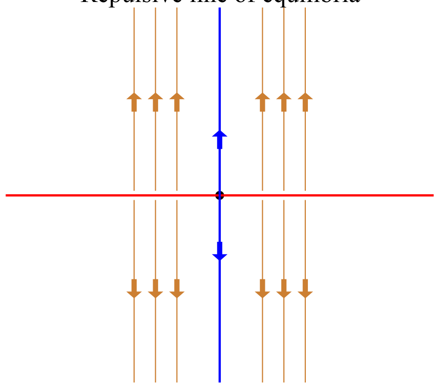


- Linear analysis alone cannot determine the correct picture.
- But we do know $(0,0)$ is unstable in the nonlinear system.

Example 7. (b)(c) Local nonlinear dynamics near $(0,0)$



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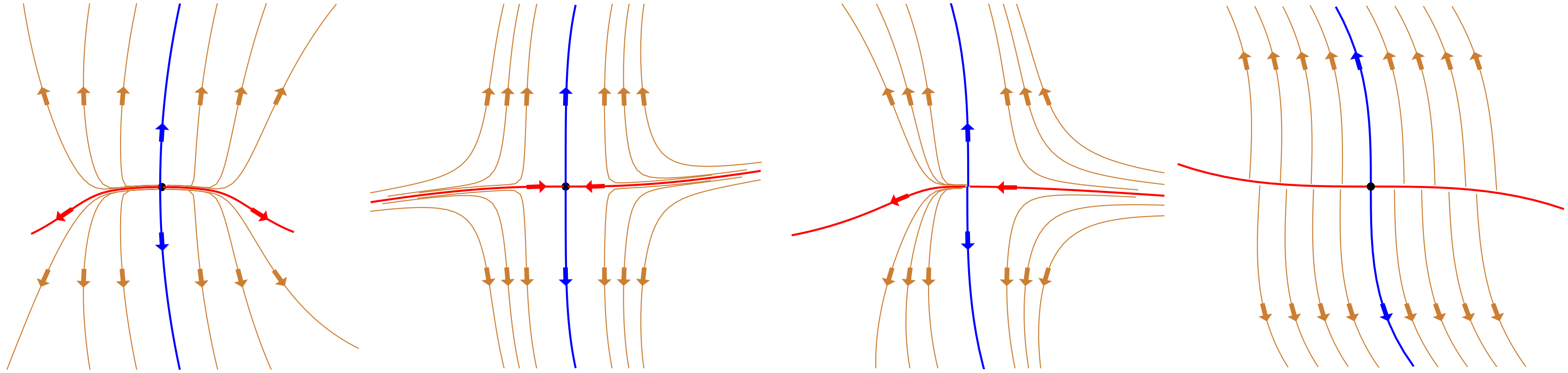

- Linear analysis alone cannot determine the correct picture.
- But we do know $(0,0)$ is unstable in the nonlinear system.
- **Reason:** since $\lambda_2 = 1 > 0$, solutions along this eigenspace will grow, with the growth rate ≈ 1, even in the nonlinear system.

Example 7. Summary.

Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

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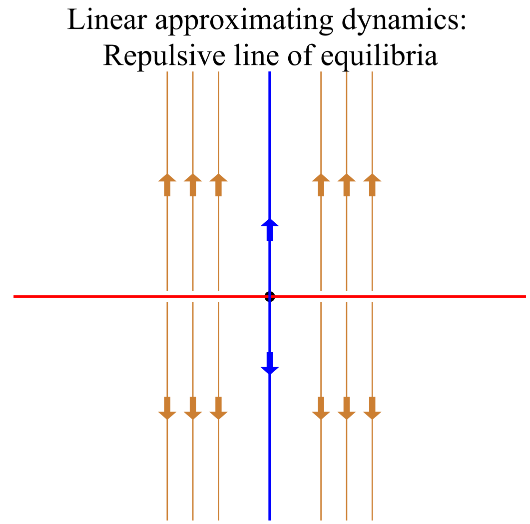

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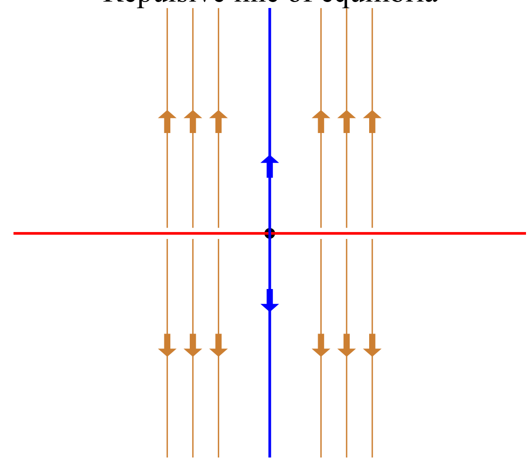

- Since one of the eigenvalues is > 0,
  the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0,0)$.

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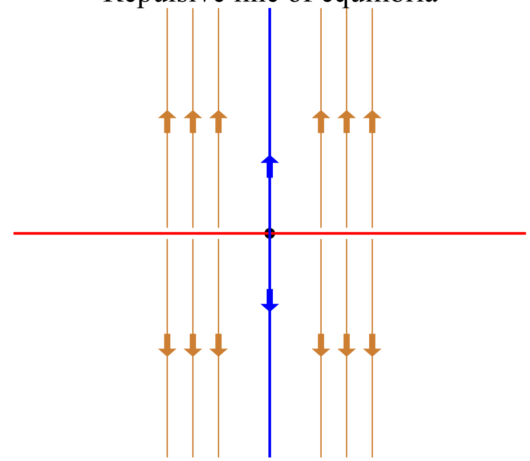


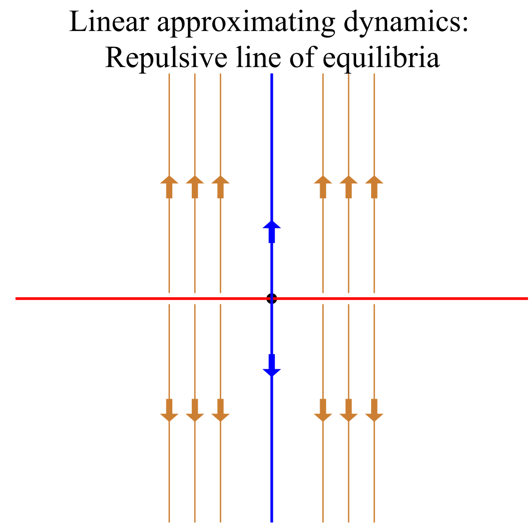
- Since one of the eigenvalues is > 0,
  the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0, 0)$.
- Since one of the eigenvalues is $= 0$ (neutral),
  the linear aprroximation $\not\Rightarrow$ the nonlinear local phase portrait.

Example 7. Summary.

Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

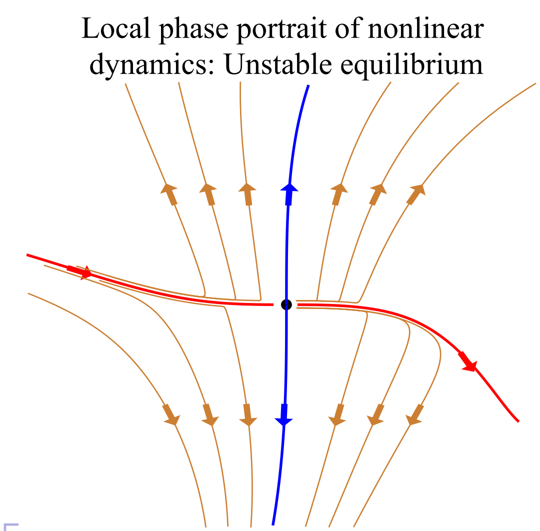
$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$



- Since one of the eigenvalues is > 0, the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0,0)$.
- Since one of the eigenvalues is $= 0$ (neutral), the linear aprroximation $\not\Rightarrow$ the nonlinear local phase portrait.

- Advanced *nonlinear* tools (center manifolds, ...) can give the local phase portrait of the nonlinear system near $(0,0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2 + \frac{1}{2} x_1^3 \end{bmatrix}$$



Example 8

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = x - z^2 \\ \dfrac{dy}{dt} = 3(-y+2z^2) \\ \dfrac{dz}{dt} = 2(-z+x^2) \end{cases}$$

(a) Find all equilibria.

(b) Give the linear approximating system near each equilibrium.

(c) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

Solution (a)

$$\begin{cases} ① \ x - z^2 = 0 \\ ② \ 3(-y+2z^2) = 0 \\ ③ \ 2(-z+x^2) = 0 \end{cases}$$

③ $\Rightarrow$ ④ $z = x^2$ substitute ④ in ① $\Rightarrow x - x^4 = 0$

$\Rightarrow x(1-x^3) = 0$

$\Rightarrow x = 0$, or $x = 1$.

- If $x = 0$, then ④ $\Rightarrow z = 0$, ② $\Rightarrow y = 2z^2 = 0$. Hence, $(x,y,z) = (0,0,0)$
- If $x = 1$, then ④ $\Rightarrow z = 1$, ② $\Rightarrow y = 2z^2 = 2$. Hence, $(x,y,z) = (1,2,1)$

Answer to (a) The equilibria are $(x,y,z) = (0,0,0)$, $(x,y,z) = (1,2,1)$.

$$(*)\begin{cases} \dfrac{dx}{dt} = f(x,y,z) = x - z^2 \\ \dfrac{dy}{dt} = g(x,y,z) = -3y + 6z^2 \\ \dfrac{dz}{dt} = h(x,y,z) = -2z + 2x^2 \end{cases}$$

- Jacobian Matrix

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2z \\ 0 & -3 & 12z \\ 4x & 0 & -2 \end{bmatrix}.$$

At equilibrium $(x,y,z)=(0,0,0)$:

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

At equilibrium $(x,y,z)=(1,2,1)$:

$$J = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 12 \\ 4 & 0 & -2 \end{bmatrix}.$$

(b)(c) for the equilibrium (0,0,0)

Jacobian $J = \begin{bmatrix} 1 & 0 & -2z \\ 0 & -3 & 12z \\ 4x & 0 & -2 \end{bmatrix}$.

At $(x, y, z) = (0, 0, 0)$:

$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

The Linear Approximating System Near (0,0,0)

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- The eigenvalues of the coefficient matrix:

$\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = -2$

$\lambda_1 > 0$

⇙

- The equilibrium (0,0,0) is unstable.

(b) (c) for the equilibrium (1, 2, 1)

$$J = \begin{bmatrix} 1 & 0 & -2z \\ 0 & -3 & 12z \\ 4x & 0 & -2 \end{bmatrix}.$$

At $(x, y, z) = (1, 2, 1)$:

$$J = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 12 \\ 4 & 0 & -2 \end{bmatrix}.$$

The Linear Approximating System Near (1, 2, 1)

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 12 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \\ z-1 \end{bmatrix}$$

- The eigenvalues of the coefficient matrix:

$$\det\begin{bmatrix} 1-\lambda & 0 & -2 \\ 0 & -3-\lambda & 12 \\ 4 & 0 & -2-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda)(-2-\lambda) - (-2)(-3-\lambda)(4)$$

$$= (-3-\lambda)\left[(1-\lambda)(-2-\lambda) - (-2)(4)\right]$$

$$= -(\lambda+3)(\lambda^2+\lambda+6).$$

$$\Rightarrow \lambda_1 = -3, \quad \lambda_{2,3} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{23}\, i$$

All have real parts < 0

- The equilibrium (1, 2, 1) is asymptotically stable.