2D Homogeneous Linear Systems with Constant Coefficients
— perturbed systems

Xu-Yan Chen

Recall basics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Eigenvalues & (generalized) eigenvectors of A
$\Rightarrow$ solution formulas, dynamics, stability/instability,....

Recall basics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Eigenvalues & (generalized) eigenvectors of A
$\Rightarrow$ solution formulas, dynamics, stability/instability,....

$$\left.\begin{array}{l}\text{Negative eigenvalues } \lambda < 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda < 0\end{array}\right\} \text{help stabilization.}$$

Recall basics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Eigenvalues & (generalized) eigenvectors of A
$\Rightarrow$ solution formulas, dynamics, stability/instability,....

$$\left.\begin{array}{l}\text{Negative eigenvalues } \lambda < 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda < 0\end{array}\right\} \text{help stabilization.}$$

$$\left.\begin{array}{l}\text{Positive eigenvalues } \lambda > 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda > 0\end{array}\right\} \text{imply instability.}$$

Recall basics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Eigenvalues & (generalized) eigenvectors of A
$\Rightarrow$ solution formulas, dynamics, stability/instability,....

$$\left.\begin{array}{l}\text{Negative eigenvalues } \lambda < 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda < 0\end{array}\right\} \text{help stabilization.}$$

$$\left.\begin{array}{l}\text{Zero eigenvalues } \lambda = 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda = 0\end{array}\right\} \text{are ``neutral''.}$$

$$\left.\begin{array}{l}\text{Positive eigenvalues } \lambda > 0 \\ \text{Complex eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda > 0\end{array}\right\} \text{imply instability.}$$

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}}) \quad \text{where } B \approx A.$

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ where $B \approx A$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ where $B \approx A$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?
- Are the dynamics of $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & that of $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ essentially the same, when B is almost equal to A?

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}}) \quad \text{where } B \approx A$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?
- Are the dynamics of $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & that of $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ essentially the same, when B is almost equal to A?
- Or, will the slightly perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ behave in ways completely different from $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}}) \quad \text{where } B \approx A$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?
- Are the dynamics of $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & that of $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ essentially the same, when B is almost equal to A?
- Or, will the slightly perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ behave in ways completely different from $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?

Questions Concerning "Structural Stability":

Suppose we have solved $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & have found the stability/instability of the equilibrium $\vec{\mathbf{x}} = \vec{\mathbf{a}}$.

Linear perturbations: Change matrix A a little bit & consider $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}}) \quad$ where $B \approx A$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?
- Are the dynamics of $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & that of $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ essentially the same, when B is almost equal to A?
- Or, will the slightly perturbed system $\vec{\mathbf{x}}\,' = B(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ behave in ways completely different from $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?

Nonlinear perturbations: Consider $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, where $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$.

- What can we tell about the perturbed system $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$, by using only the info about $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$?
- Are the dynamics of $\vec{\mathbf{x}}\,' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ & that of $\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ essentially the same? Or, will they be markedly different?

Example 1.

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$$

Eigenvalues & eigenvectors:

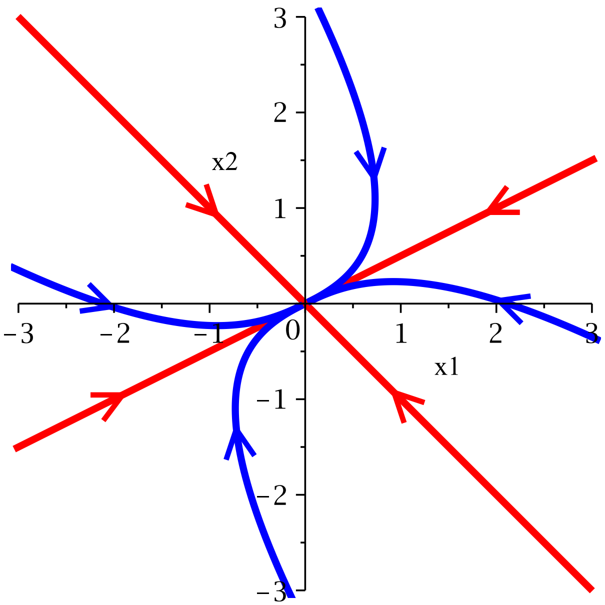
$$\lambda_1 = -2, \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Original Linear Dynamics: Attractive Improper Node



Example 1.

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$

Eigenvalues & eigenvectors:

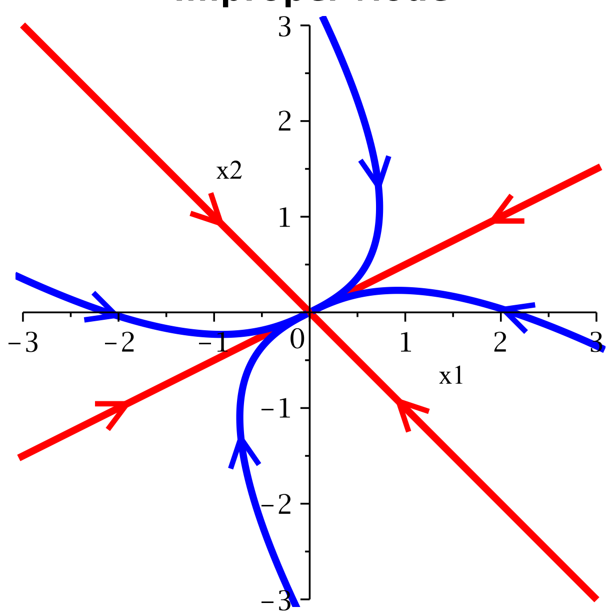
$\lambda_1 = -2, \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

General solutions:

$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Original Linear Dynamics: Attractive Improper Node



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -2.98 & 1.98 \\ 0.97 & -4.01 \end{bmatrix}$

Example 1.

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$

Eigenvalues & eigenvectors:

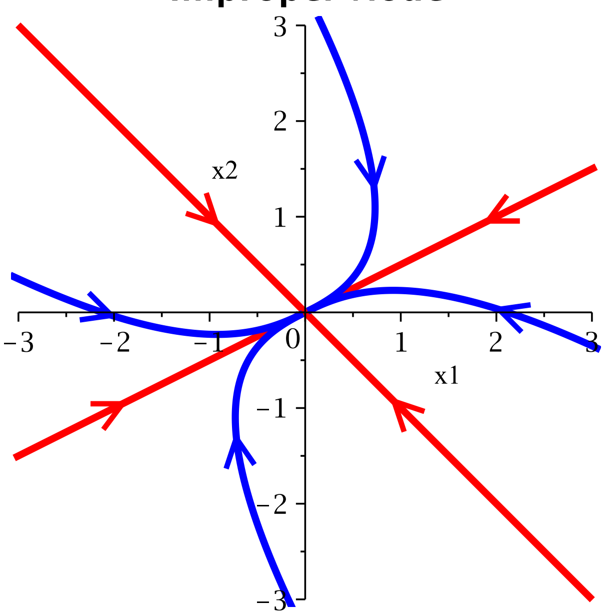
$\lambda_1 = -2, \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

General solutions:

$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Original Linear Dynamics: Attractive Improper Node



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -2.98 & 1.98 \\ 0.97 & -4.01 \end{bmatrix}$

Eigenvalues & eigenvectors:

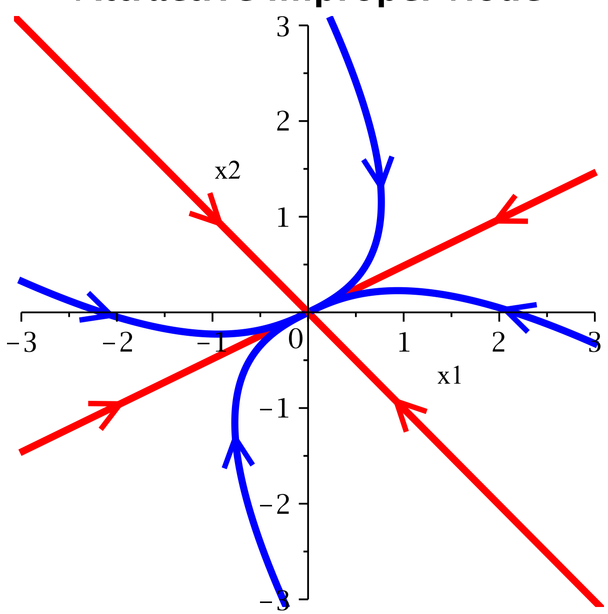
$\lambda_1 \approx -2.017, \vec{\mathbf{u}}_1 \approx \begin{bmatrix} 2.055 \\ 1 \end{bmatrix}$

$\lambda_2 \approx -4.973, \vec{\mathbf{u}}_2 \approx \begin{bmatrix} -0.993 \\ 1 \end{bmatrix}$

General solutions:

$\vec{\mathbf{x}}(t) = C_1 e^{-2.017t} \begin{bmatrix} 2.055 \\ 1 \end{bmatrix} + C_2 e^{-5.973t} \begin{bmatrix} -0.993 \\ 1 \end{bmatrix}$

Perturbed Linear Dynamics: Attractive Improper Node



Example 2 (a).

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$$

Eigenvalues & eigenvectors:

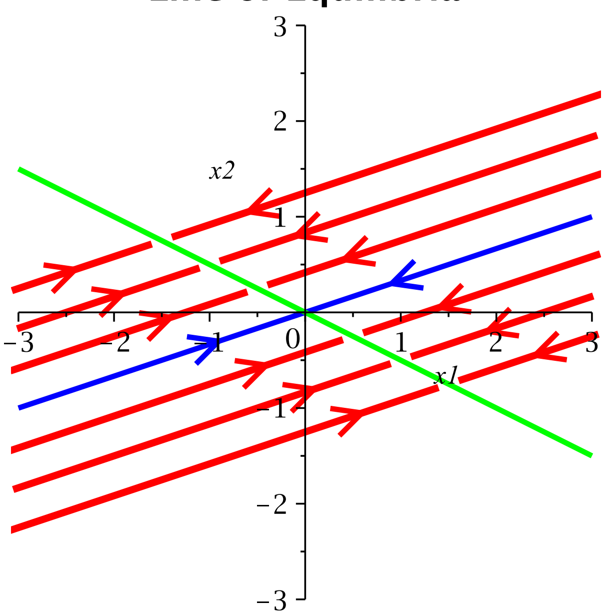
$$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Original Linear Dyanmics: Attractive Line of Equilibria



Example 2 (a).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$

$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -3.09 & -5.83 \\ -0.98 & -2.01 \end{bmatrix}$

Eigenvalues & eigenvectors:

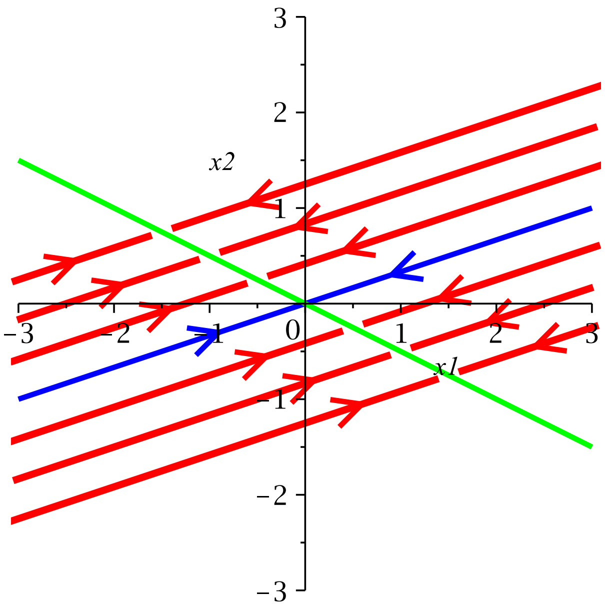
$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Original Linear Dyanmics: Attractive Line of Equilibria



Example 2 (a).

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$$

Eigenvalues & eigenvectors:

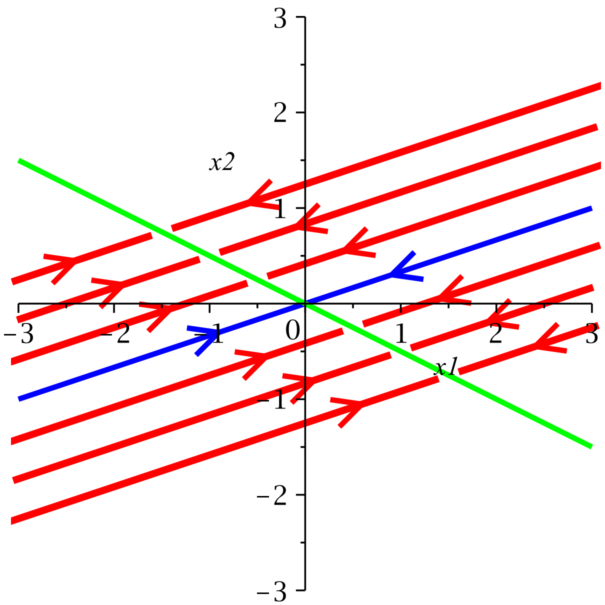
$$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Original Linear Dyanmics: Attractive Line of Equilibria



$$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -3.09 & -5.83 \\ -0.98 & -2.01 \end{bmatrix}$$

Eigenvalues & eigenvectors:

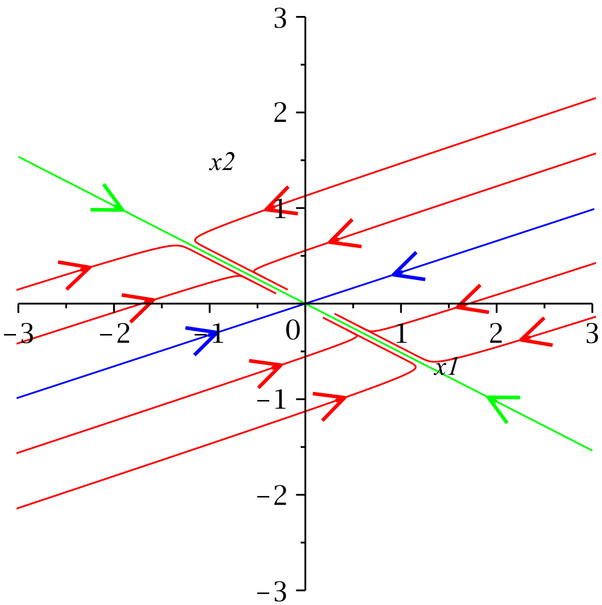
$$\lambda_1 \approx -0.1, \vec{\mathbf{u}}_1 \approx \begin{bmatrix} -1.95 \\ 1 \end{bmatrix}$$

$$\lambda_2 \approx -5.0, \vec{\mathbf{u}}_2 \approx \begin{bmatrix} 3.05 \\ 1 \end{bmatrix}$$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-0.1t} \begin{bmatrix} -1.95 \\ 1 \end{bmatrix} + C_2 e^{-5.0t} \begin{bmatrix} 3.05 \\ 1 \end{bmatrix}$$

Perturbed Linear Dynamics: Attractive Improper Node



Example 2 (b).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}, \; A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$

Eigenvalues & eigenvectors:

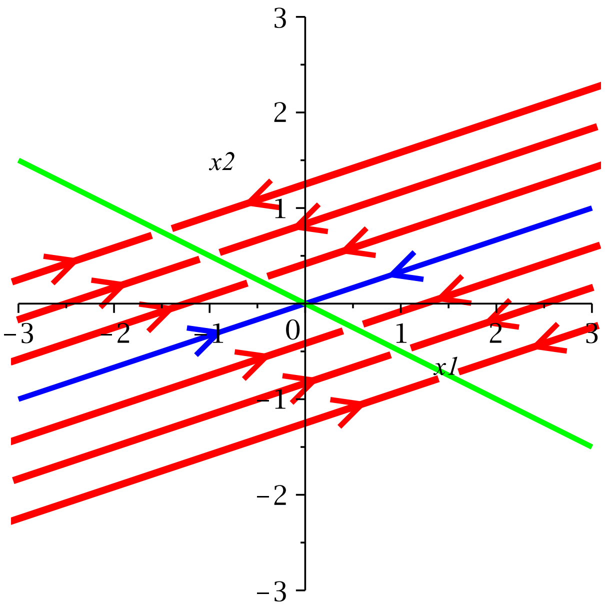
$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

General solutions:

$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Original Linear Dyanmics: Attractive Line of Equilibria



Example 2 (b).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$

Eigenvalues & eigenvectors:

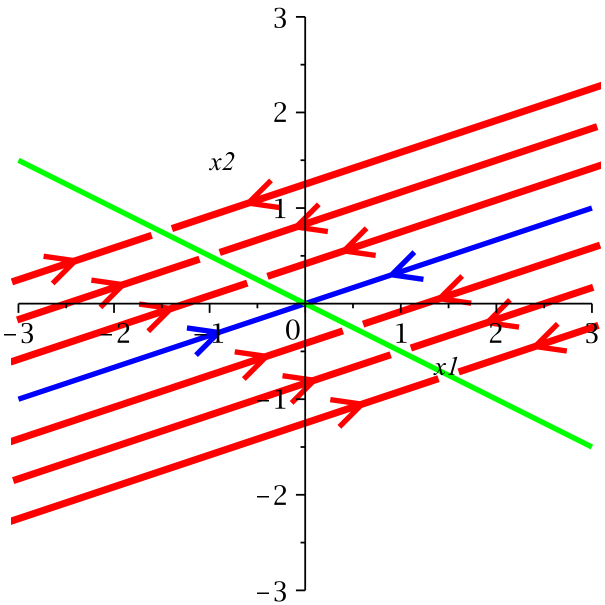
$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Original Linear Dyanmics: Attractive Line of Equilibria



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -3.09 & -6.21 \\ -0.96 & -1.76 \end{bmatrix}$

Example 2 (b).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$

Eigenvalues & eigenvectors:

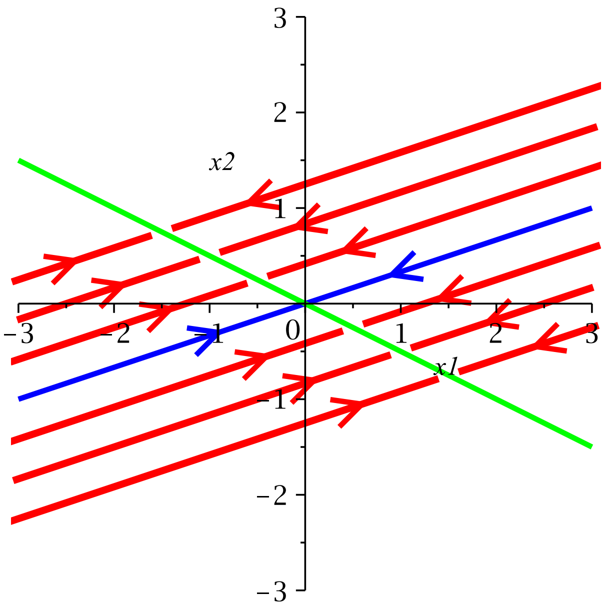
$\lambda_1 = 0, \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Original Linear Dyanmics: Attractive Line of Equilibria



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} -3.09 & -6.21 \\ -0.96 & -1.76 \end{bmatrix}$

Eigenvalues & eigenvectors:

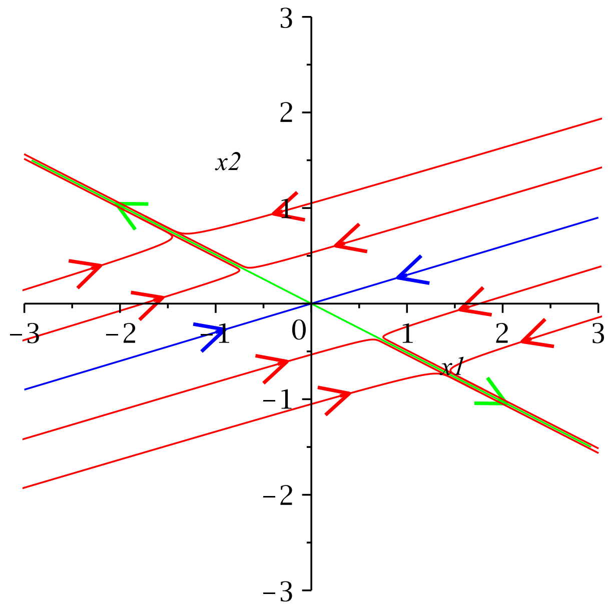
$\lambda_1 \approx 0.1, \vec{\mathbf{u}}_1 \approx \begin{bmatrix} -1.95 \\ 1 \end{bmatrix}$

$\lambda_2 \approx -4.95, \vec{\mathbf{u}}_2 \approx \begin{bmatrix} 3.33 \\ 1 \end{bmatrix}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{0.1t} \begin{bmatrix} -1.95 \\ 1 \end{bmatrix} + C_2 e^{-4.950t} \begin{bmatrix} 3.22 \\ 1 \end{bmatrix}$$

Another Perturbed Linear Dynamics: Saddle

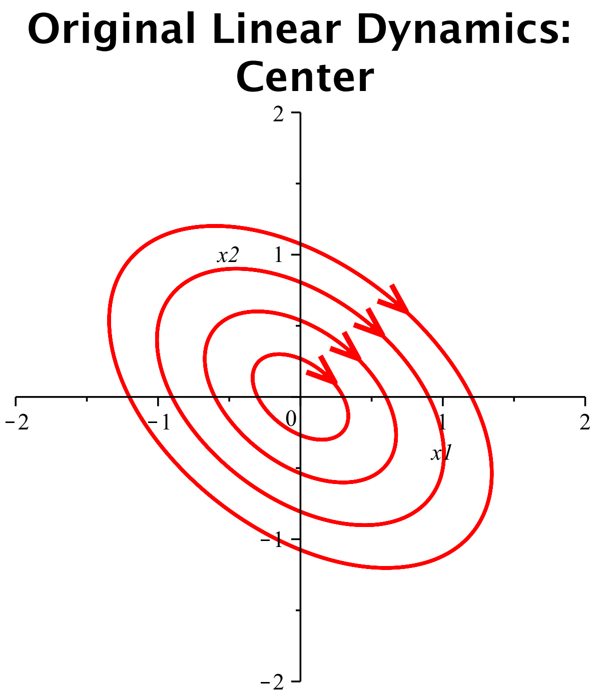


Example 3 (a).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$

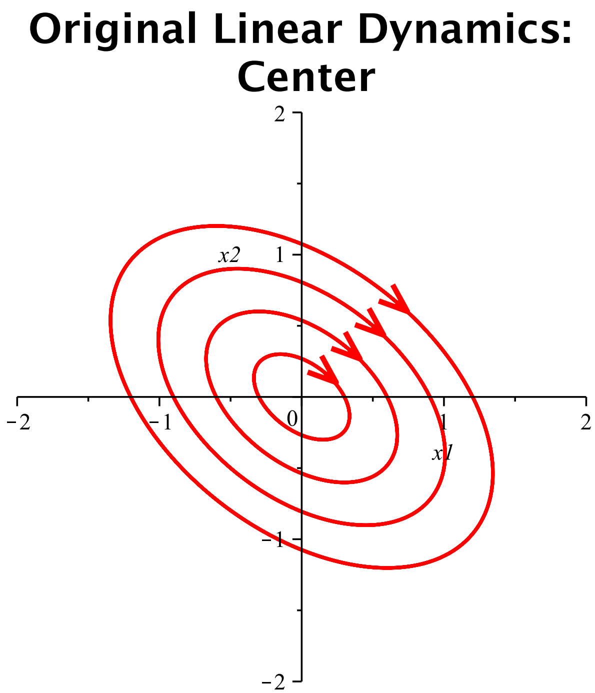


Example 3 (a).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} 1.04 & 2.51 \\ -2.01 & -0.95 \end{bmatrix}$

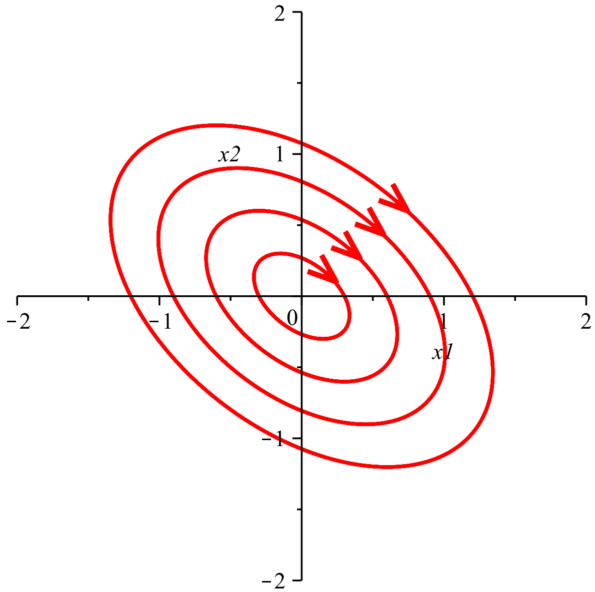
Example 3 (a).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$

Original Linear Dynamics: Center

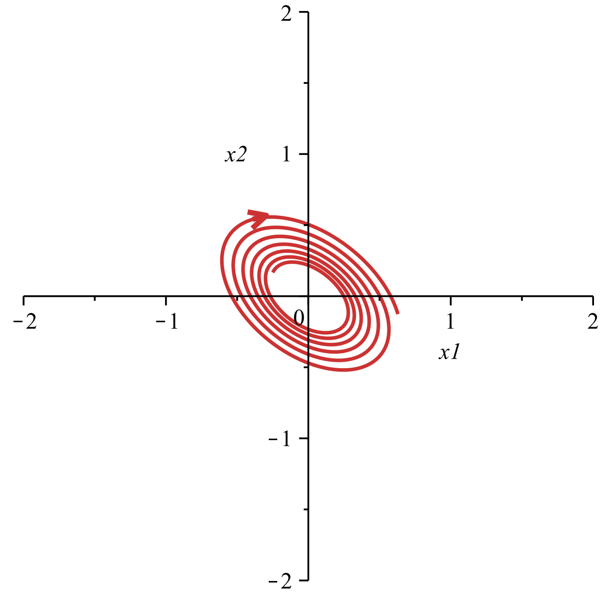


$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} 1.04 & 2.51 \\ -2.01 & -0.95 \end{bmatrix}$

Eigenvalues:

$\lambda_{1,2} \approx 0.045 \pm 2.014\, i$

Perturbed Linear Dynamics: Repulsive Focus

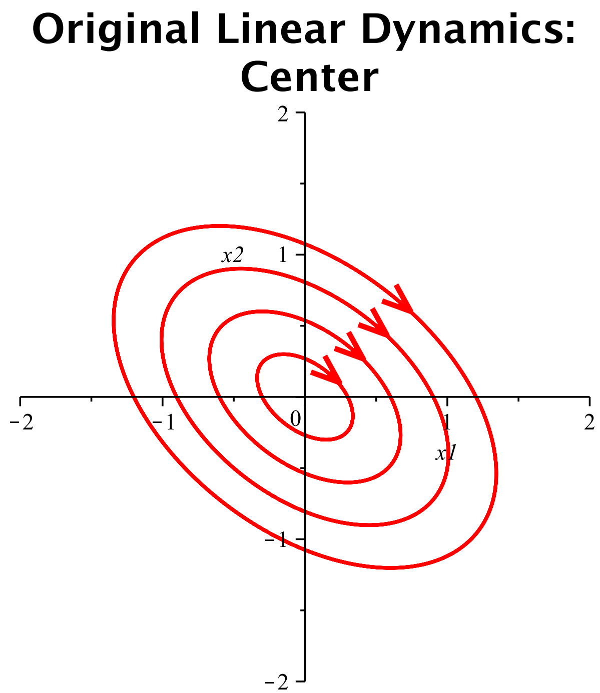


Example 3 (b).

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}, \; A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$



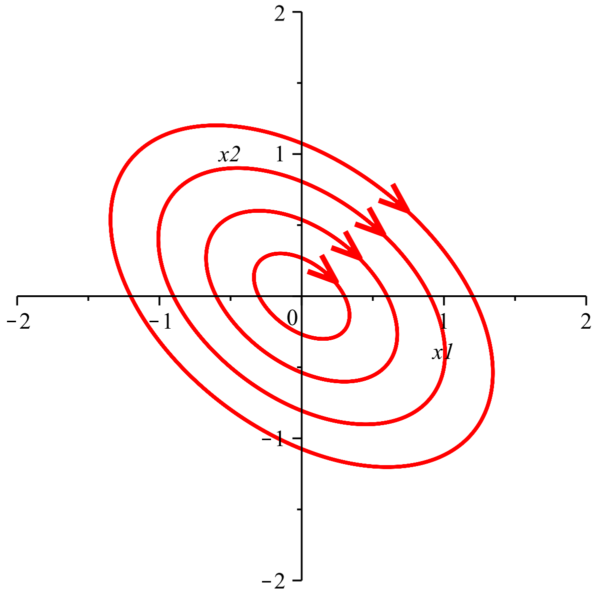
Example 3 (b).

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$

Original Linear Dynamics: Center



$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} 0.97 & 2.51 \\ -1.99 & -1.02 \end{bmatrix}$

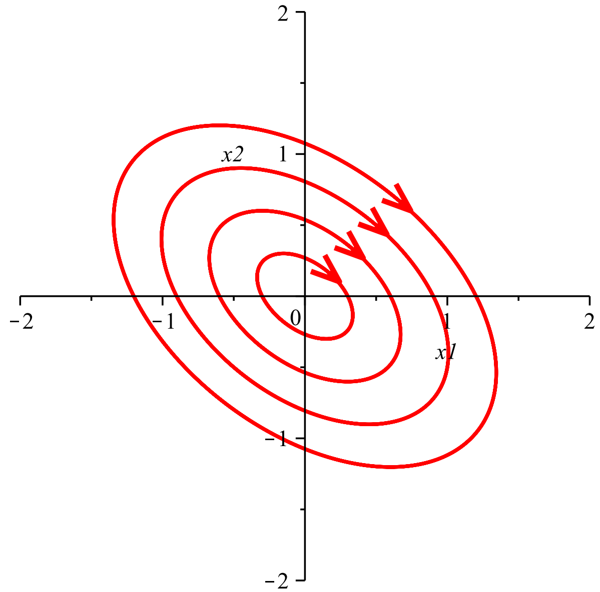
Example 3 (b).

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}},\ A = \begin{bmatrix} 1 & 5/2 \\ -2 & -1 \end{bmatrix}$$

Eigenvalues:

$\lambda_{1,2} = \pm 2i$

Original Linear Dynamics: Center

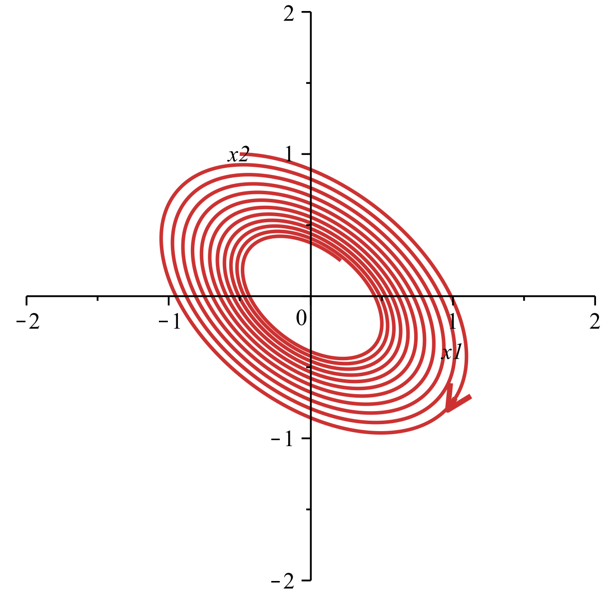


$$\vec{\mathbf{x}}' = B\vec{\mathbf{x}},\ B = \begin{bmatrix} 0.97 & 2.51 \\ -1.99 & -1.02 \end{bmatrix}$$

Eigenvalues:

$\lambda_{1,2} \approx -0.025 \pm 2.001\, i$

Another Perturbed Linear Dynamics: Attractive Focus



The Morals of the Story:

"Neutral" eigenvalues ($\lambda = 0$ or $\mathrm{Re}\,\lambda = 0$)
are the sources of **structural instability**.

If the system has neutral eigenvalues, a tiny change in the diff eqs may alter the phase portrait completely.

Other "non-neutral" eigenvalues
give **structural stable** dynamics.

If the system has no neutral eigenvalues, small changes in diff eqs will not change the dynamics radically & will only give an equivalent phase portrait.

Linear Perturbation Theorems:

Theorem 1. If A has a neutral eigenvalue ($\lambda = 0$ or $\mathrm{Re}\,\lambda = 0$), then the dynamics are sensitive to the coeff. perturbations.

In this case, for some matrices $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	may be radically different from	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Linear Perturbation Theorems:

Theorem 1. If A has a neutral eigenvalue ($\lambda = 0$ or $\mathrm{Re}\,\lambda = 0$), then the dynamics are sensitive to the coeff. perturbations.

In this case, for some matrices $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	may be radically different from	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Theorem 2. If all eigenvalues of A have nonzero real parts, then the dynamics are robust to the coefficient perturbations.

In this case, for any matrix $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	are essentially equivalent to	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Linear Perturbation Theorems:

Theorem 1. If A has a neutral eigenvalue ($\lambda = 0$ or $\operatorname{Re}\lambda = 0$), then the dynamics are sensitive to the coeff. perturbations.

In this case, for some matrices $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	may be radically different from	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Theorem 2. If all eigenvalues of A have nonzero real parts, then the dynamics are robust to the coefficient perturbations.

In this case, for any matrix $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	are essentially equivalent to	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Theorem 3. If all eigenvalues of A have real parts < 0, then the equilibrium is asymp. stable not only for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, but is also asymp. stable for $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$ for all $B \approx A$.

Linear Perturbation Theorems:

Theorem 1. If A has a neutral eigenvalue ($\lambda = 0$ or $\mathrm{Re}\,\lambda = 0$), then the dynamics are sensitive to the coeff. perturbations.

In this case, for some matrices $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	may be radically different from	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Theorem 2. If all eigenvalues of A have nonzero real parts, then the dynamics are robust to the coefficient perturbations.

In this case, for any matrix $B \approx A$,

the dynamics of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$	are essentially equivalent to	the dynamics of $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$

Theorem 3. If all eigenvalues of A have real parts < 0, then the equilibrium is asymp. stable not only for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, but is also asymp. stable for $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$ for all $B \approx A$.

Theorem 4. If at least one eigenvalue of A has real part > 0, then the equilibrium is unstable not only for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, but is also unstable for $\vec{\mathbf{x}}' = B\vec{\mathbf{x}}$ for all $B \approx A$.

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	are essentially equivalent to	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	are essentially equivalent to	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If A has a neutral eigenvalue, then the local dynamics are sensitive to nonlin. perturbations.

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	are essentially equivalent to	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If A has a neutral eigenvalue, then the local dynamics are sensitive to nonlin. perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	may be radically different from	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	are essentially equivalent to	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If A has a neutral eigenvalue, then the local dynamics are sensitive to nonlin. perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	may be radically different from	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If all eigenvalues of A have real parts < 0, then $\vec{\mathbf{x}} = \vec{\mathbf{a}}$ is asymp. stable for $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ as long as $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$.

Nonlinear Perturbation Theorems:

If all eigenvalues of A have nonzero real parts, then the local dynamics are robust to nonlinear perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	are essentially equivalent to	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If A has a neutral eigenvalue, then the local dynamics are sensitive to nonlin. perturbations.

In this case, for $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ for $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$,

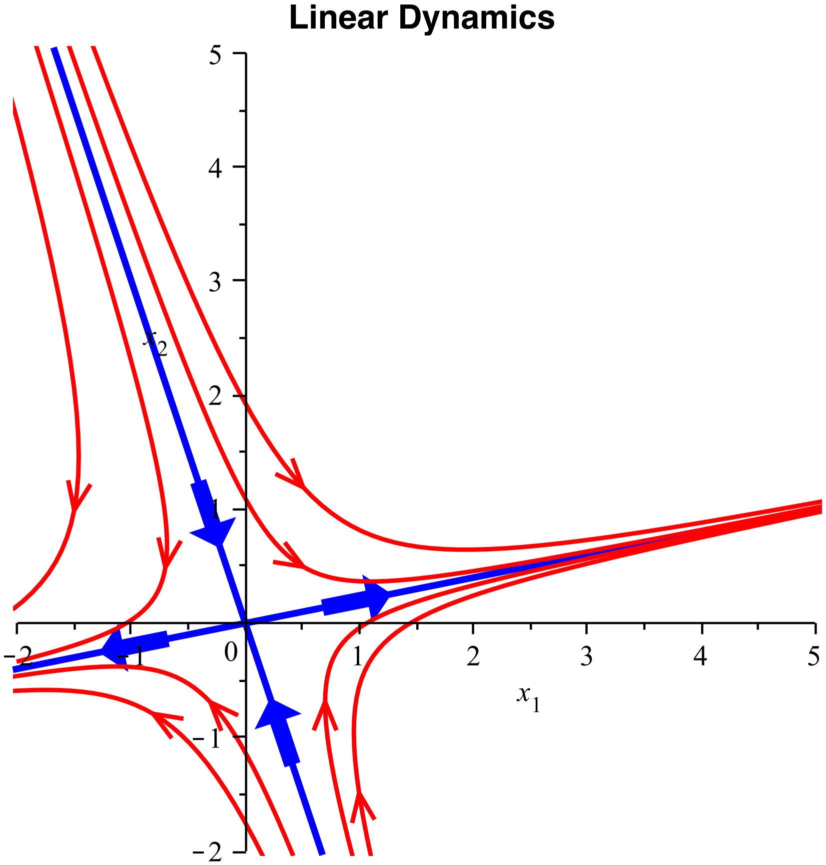
the local dynamics of $\vec{\mathbf{x}}' = A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$	may be radically different from	the local dynamics of $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$

If all eigenvalues of A have real parts < 0, then $\vec{\mathbf{x}} = \vec{\mathbf{a}}$ is asymp. stable for $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ as long as $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$.

If at least one eigenvalue of A has real part > 0, then $\vec{\mathbf{x}} = \vec{\mathbf{a}}$ is unstable for $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}})$ as long as $\vec{\mathbf{f}}(\vec{\mathbf{x}}) \approx A(\vec{\mathbf{x}} - \vec{\mathbf{a}})$ near $\vec{\mathbf{x}} \approx \vec{\mathbf{a}}$.

Example 4. (Nonlinear perturbation)

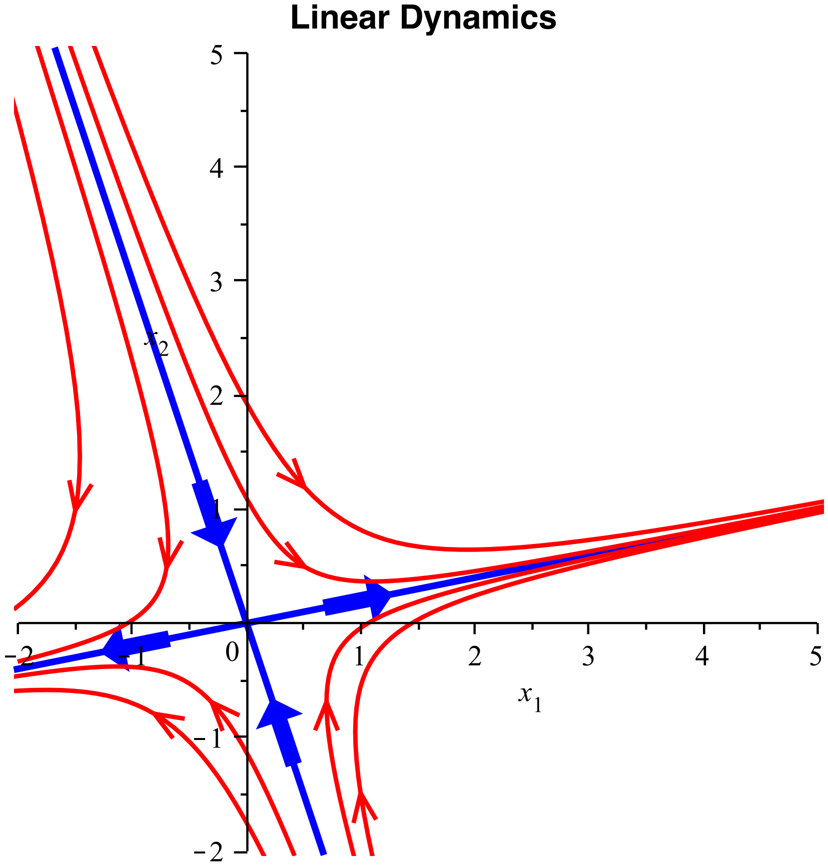
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \\ 3x_1 - 9x_2 \end{bmatrix}$$



Example 4. (Nonlinear perturbation)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \\ 3x_1 - 9x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 - 5x_1x_2 \\ 3x_1 - 9x_2 + 9x_2^2 - 3x_2^3 \end{bmatrix}$$

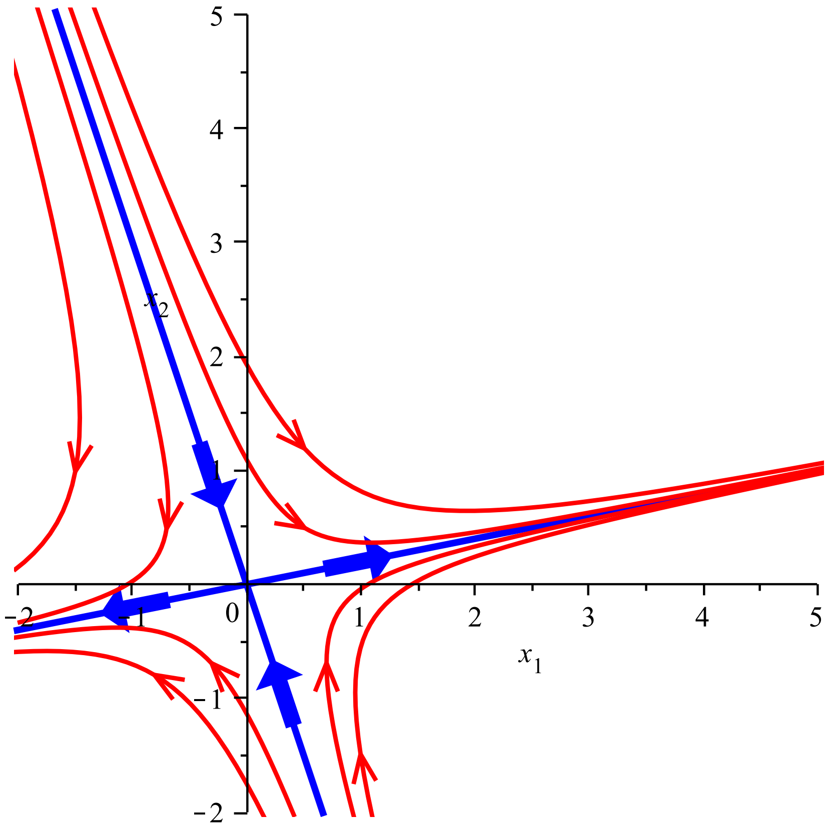


Example 4. (Nonlinear perturbation)

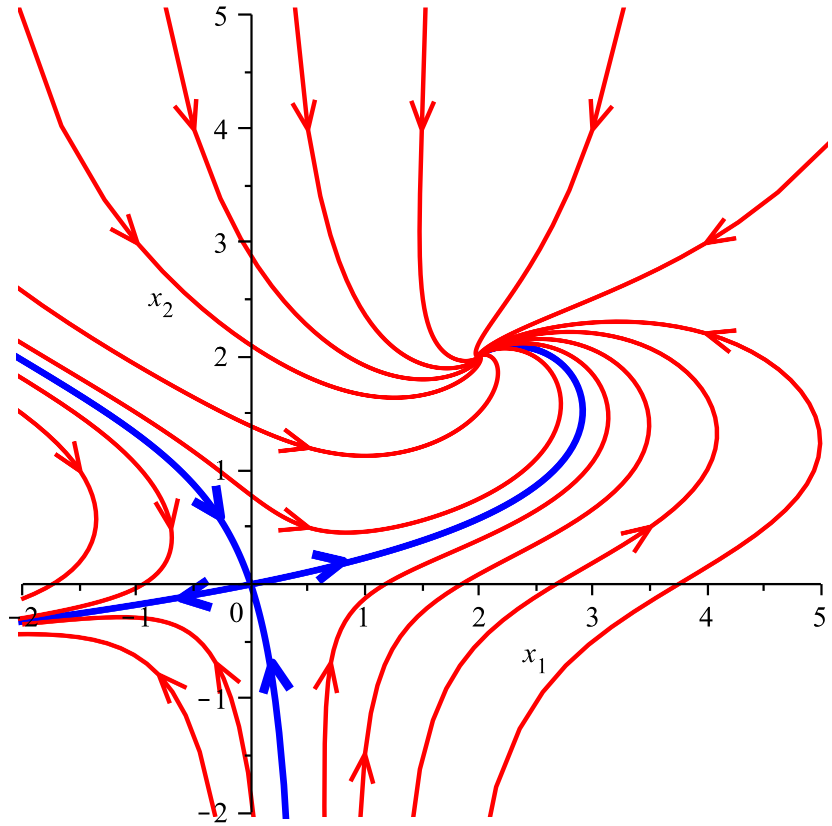
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \\ 3x_1 - 9x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \ -5x_1x_2 \\ 3x_1 - 9x_2 \ +9x_2^2 - 3x_2^3 \end{bmatrix}$$

Linear Dynamics

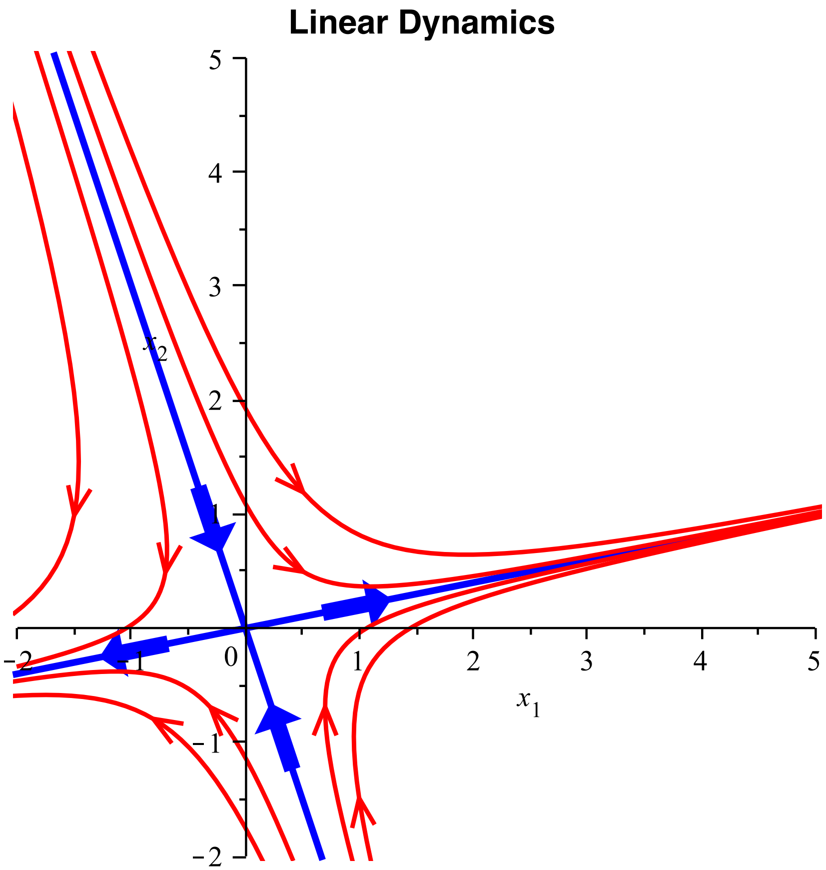


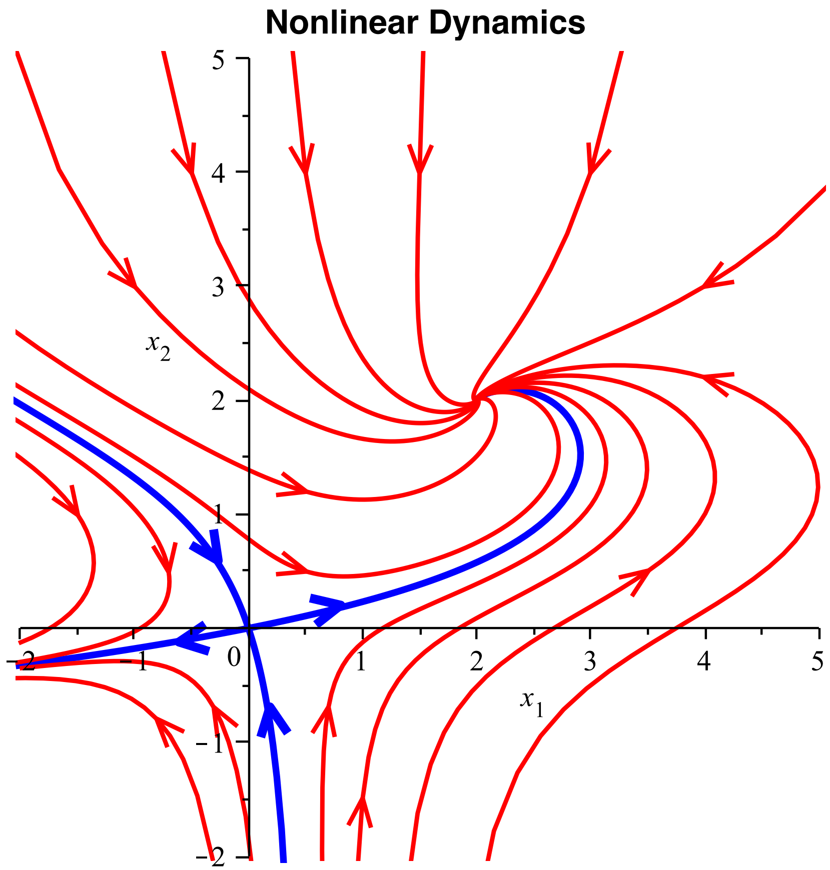
Nonlinear Dynamics



Example 4. (Nonlinear perturbation)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \\ 3x_1 - 9x_2 \end{bmatrix} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \; -5x_1x_2 \\ 3x_1 - 9x_2 \; +9x_2^2 - 3x_2^3 \end{bmatrix}$$

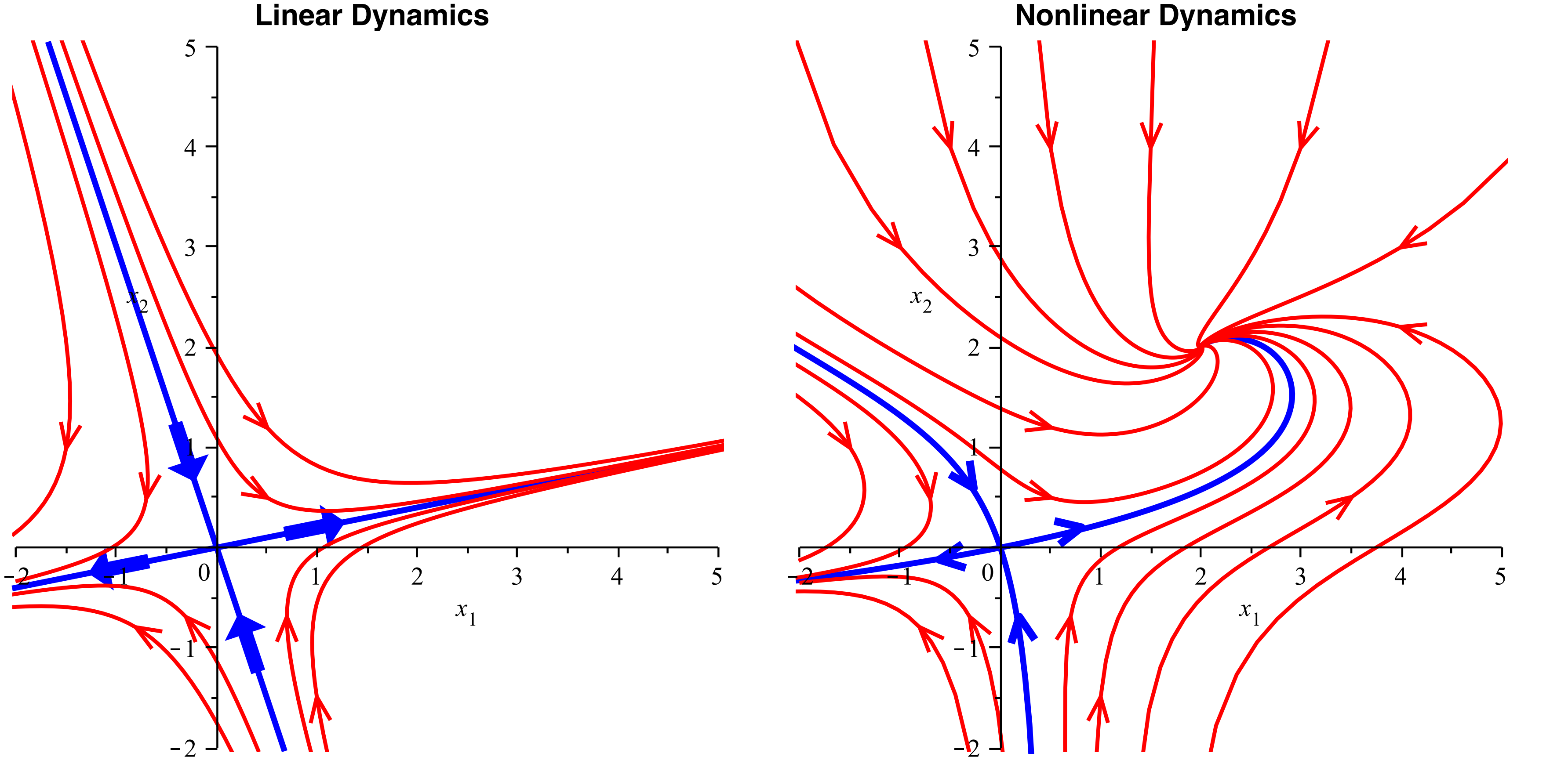




Near the equilibrium $(0,0)$**:** The perturbation terms are almost negligible & the two phase portraits are **locally** equivalent.

Example 4. (Nonlinear perturbation)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 \\ 3x_1 - 9x_2 \end{bmatrix} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 - 5x_1x_2 \\ 3x_1 - 9x_2 + 9x_2^2 - 3x_2^3 \end{bmatrix}$$



Near the equilibrium $(0,0)$: The perturbation terms are almost negligible & the two phase portraits are **locally** equivalent.

Far from $(0,0)$: The red terms are no longer small. The two phase portraits are globally non-equivalent.

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx$
    $f_1(a_1, a_2) + \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - $f_2(x_1, x_2) \approx$
    $f_2(a_1, a_2) + \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - $f_2(x_1, x_2) \approx \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - $f_2(x_1, x_2) \approx \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - The linear approx system near equilibrium (a_1, a_2) is:

$$x_1' = \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$$

$$x_2' = \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$$

The Linear Approximating System near an equilibrium

$$\vec{\mathbf{x}}\,' = \vec{\mathbf{f}}(\vec{\mathbf{x}}) \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$
- Near an equilibria $\vec{\mathbf{a}} = (a_1, a_2)$, take a linear approximation:
  - $f_1(x_1, x_2) \approx \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - $f_2(x_1, x_2) \approx \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$
  - The linear approx system near equilibrium (a_1, a_2) is:

$$x_1' = \frac{\partial f_1}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a_1, a_2)(x_2 - a_2)$$
$$x_2' = \frac{\partial f_2}{\partial x_1}(a_1, a_2)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a_1, a_2)(x_2 - a_2)$$

  - Or, equivalently, the linear approx system is:

$$\vec{\mathbf{x}}\,' = J(\vec{\mathbf{x}} - \vec{\mathbf{a}}),$$

where $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a_1, a_2) & \frac{\partial f_1}{\partial x_2}(a_1, a_2) \\ \frac{\partial f_2}{\partial x_1}(a_1, a_2) & \frac{\partial f_2}{\partial x_2}(a_1, a_2) \end{bmatrix}$ is the Jacobian matrix.

Example 5.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find all equilibria.
- For each equilibrium, give the linear approximating system near it.
- Sketch the phase portrait of the linear approximating system.
- Determine whether each equilibrium is stable or unstable with respect to the nonlinear system.

Example 5. Find equilibria.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

Example 5. Find equilibria.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

Example 5. Find equilibria.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad -x_1 - x_2 = 0 \\ (2) \quad x_1 - 7x_2 + x_2^2 - 3x_1x_2 = 0 \end{cases}$$

Example 5. Find equilibria.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad -x_1 - x_2 = 0 \\ (2) \quad x_1 - 7x_2 + x_2^2 - 3x_1x_2 = 0 \end{cases}$$

From (1), $x_2 = -x_1$.
Substitute this in (2): $8x_1 + 4x_1^2 = 0 \Rightarrow x_1 = 0$, or $x_1 = -2$.

Example 5. Find equilibria.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad -x_1 - x_2 = 0 \\ (2) \quad x_1 - 7x_2 + x_2^2 - 3x_1x_2 = 0 \end{cases}$$

From (1), $x_2 = -x_1$.
Substitute this in (2): $8x_1 + 4x_1^2 = 0 \Rightarrow x_1 = 0$, or $x_1 = -2$.

Combined with $x_2 = -x_1$:

$\Rightarrow$ Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.

Example 5. Linear approximating system near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.

Example 5. Linear approximating system near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 - 3x_2 & -7 - 3x_1 + 2x_2 \end{bmatrix}$$

Example 5. Linear approximating system near the equilibrium $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 - 3x_2 & -7 - 3x_1 + 2x_2 \end{bmatrix}$$

- Near the equilibrium $(0, 0)$, construct a linear approx. system:

Example 5. Linear approximating system near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = -x_1 - x_2 \\ f_2(x_1,x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1,x_2) = (0,0)$, $(x_1,x_2) = (-2,2)$.
- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 - 3x_2 & -7 - 3x_1 + 2x_2 \end{bmatrix}$$

- Near the equilibrium $(0,0)$, construct a linear approx. system:
  - Evaluate J at equilibrium $(x_1,x_2) = (0,0)$: $\quad J = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix}$

Example 5. Linear approximating system near the equilibrium $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 - 3x_2 & -7 - 3x_1 + 2x_2 \end{bmatrix}$$

- Near the equilibrium $(0, 0)$, construct a linear approx. system:
  - Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $\quad J = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix}$
  - The linear approximating system near $(0, 0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 5. Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Linear approx system near $(0, 0)$: $\quad \vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$

Example 5. Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Linear approx system near $(0, 0)$: $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors: $\lambda_1 = -4 + 2\sqrt{2} < 0, \ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix},$ $\lambda_2 = -4 - 2\sqrt{2} < 0, \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix}$

Example 5. Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Linear approx system near $(0, 0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors: $$\begin{aligned} \lambda_1 &= -4 + 2\sqrt{2} < 0, \ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix}, \\ \lambda_2 &= -4 - 2\sqrt{2} < 0, \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix} \end{aligned}$$
- Thus, $(0, 0)$ is an attractive node & is asymptotically stable in the linear dynamics.

Example 5. Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = -x_1 - x_2 \\ f_2(x_1,x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0,0)$, $(x_1, x_2) = (-2, 2)$.
- Linear approx system near $(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:
$$\lambda_1 = -4 + 2\sqrt{2} < 0,\ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix},$$
$$\lambda_2 = -4 - 2\sqrt{2} < 0,\ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix}$$
- Thus, $(0,0)$ is an attractive node & is asymptotically stable in the linear dynamics.
- Since the eigenvalues are not neutral,

the nonlinear dynamics are equivalent to the linear dynamics near $(0,0)$.

Example 5. Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = -x_1 - x_2 \\ f_2(x_1,x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1,x_2) = (0,0)$, $(x_1,x_2) = (-2,2)$.
- Linear approx system near $(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors: $$\lambda_1 = -4 + 2\sqrt{2} < 0,\ \ \vec{\mathbf{u}}_1 = \begin{bmatrix} 3 + 2\sqrt{2} \\ 1 \end{bmatrix},$$ $$\lambda_2 = -4 - 2\sqrt{2} < 0,\ \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 - 2\sqrt{2} \\ 1 \end{bmatrix}$$
- Thus, $(0,0)$ is an attractive node & is asymptotically stable in the linear dynamics.
- Since the eigenvalues are not neutral, the nonlinear dynamics are equivalent to the linear dynamics near $(0,0)$.
- Equilibrium $(0,0)$ is asymptotically stable with respect to the original nonlinear system.

Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.

Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Evaluate J at $(x_1, x_2) = (-2, 2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$

Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Evaluate J at $(x_1, x_2) = (-2, 2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$
- Linear approx system near $(-2, 2)$: $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 + 2 \\ x_2 - 2 \end{bmatrix}$

Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Evaluate J at $(x_1, x_2) = (-2, 2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$
- Linear approx system near $(-2, 2)$: $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 + 2 \\ x_2 - 2 \end{bmatrix}$
- Eigenvalues & eigenvectors: $$\lambda_1 = 4, \ \vec{\mathbf{u}}_1 = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Evaluate J at $(x_1, x_2) = (-2, 2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$
- Linear approx system near $(-2, 2)$: $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 + 2 \\ x_2 - 2 \end{bmatrix}$
- Eigenvalues & eigenvectors: $\quad \lambda_1 = 4, \ \vec{\mathbf{u}}_1 = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Thus, $(-2, 2)$ is a saddle & is unstable in the linear dynamics.

Example 5. Dynamics near $(-2,2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = -x_1 - x_2 \\ f_2(x_1,x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

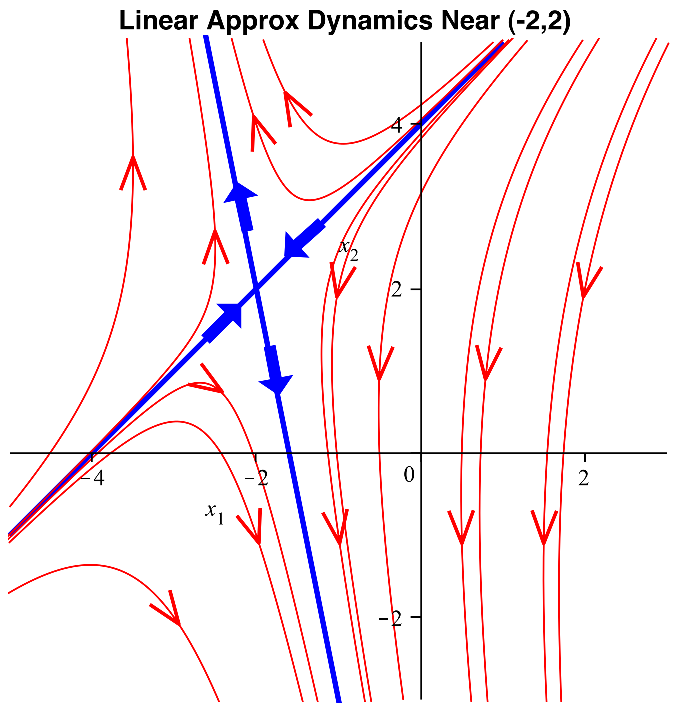
- Two equilibria: $(x_1,x_2) = (0,0)$, $(x_1,x_2) = (-2,2)$.
- Evaluate J at $(x_1,x_2) = (-2,2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$
- Linear approx system near $(-2,2)$: $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1+2 \\ x_2-2 \end{bmatrix}$
- Eigenvalues & eigenvectors: $\quad \lambda_1 = 4,\ \vec{\mathbf{u}}_1 = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2,\ \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Thus, $(-2,2)$ is a saddle & is unstable in the linear dynamics.
- Since the eigenvalues are not neutral,
the nonlinear dynamics are equivalent to the linear dynamics near $(-2,2)$.

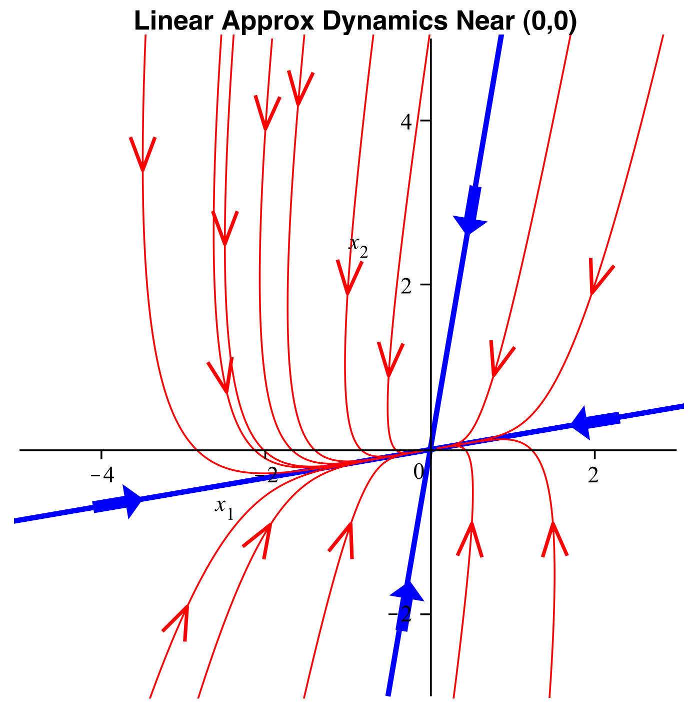
Example 5. Dynamics near $(-2, 2)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = -x_1 - x_2 \\ f_2(x_1, x_2) = x_1 - 7x_2 + x_2^2 - 3x_1x_2 \end{cases}$$

- Two equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (-2, 2)$.
- Evaluate J at $(x_1, x_2) = (-2, 2)$: $\quad J = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$
- Linear approx system near $(-2, 2)$: $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 + 2 \\ x_2 - 2 \end{bmatrix}$
- Eigenvalues & eigenvectors: $\quad \lambda_1 = 4, \ \vec{\mathbf{u}}_1 = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \ \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Thus, $(-2, 2)$ is a saddle & is unstable in the linear dynamics.
- Since the eigenvalues are not neutral,
the nonlinear dynamics are equivalent to the linear dynamics near $(-2, 2)$.
- Equilibrium $(-2, 2)$ is also a saddle with respect to the original nonlinear system & it is unstable.

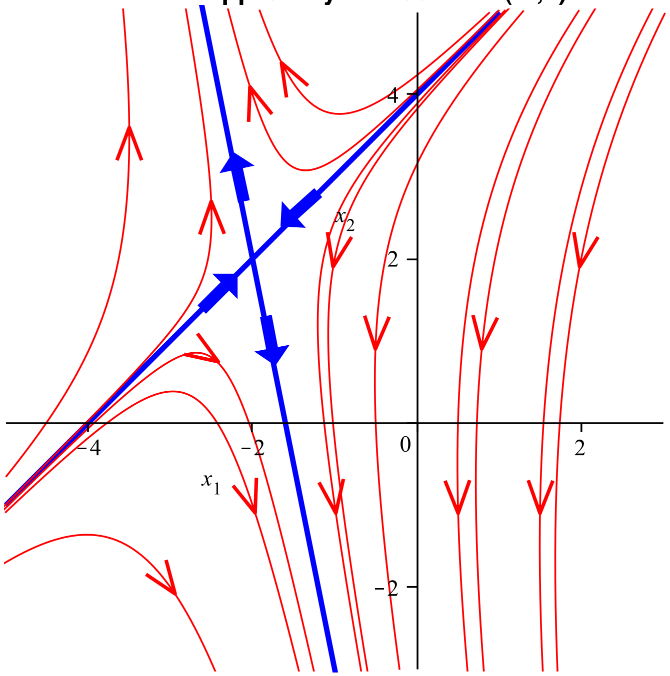
Example 5. Since all the eigenvalues are non-neutral,
Linear approx dynamics $\Rightarrow$ Nonlinear local dynamics near equilibria



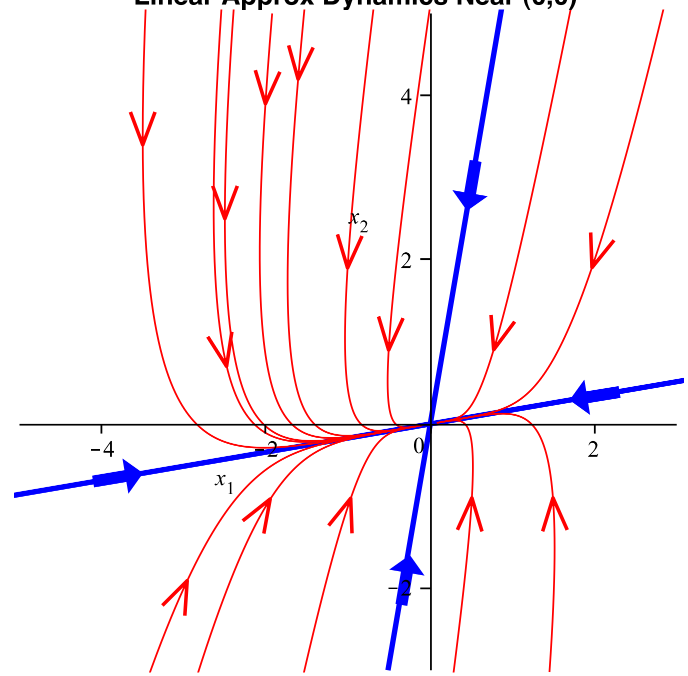


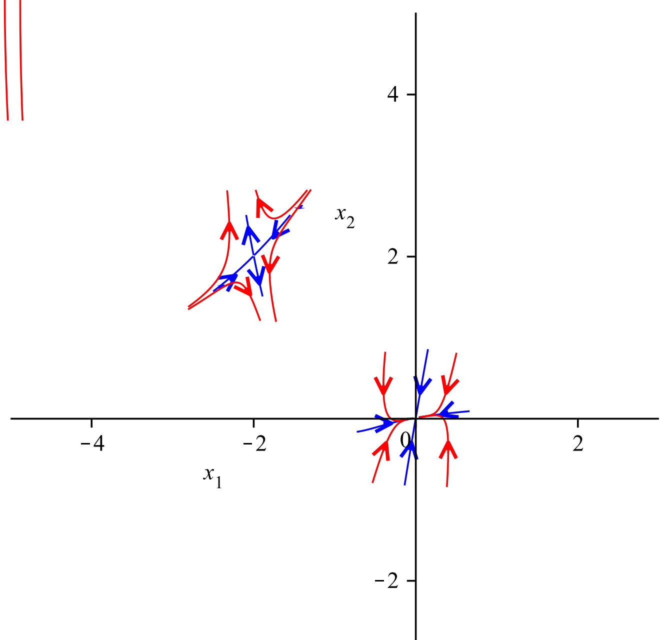
Example 5. Since all the eigenvalues are non-neutral,
Linear approx dynamics $\Rightarrow$ Nonlinear local dynamics near equilibria

Linear Approx Dynamics Near (-2,2)



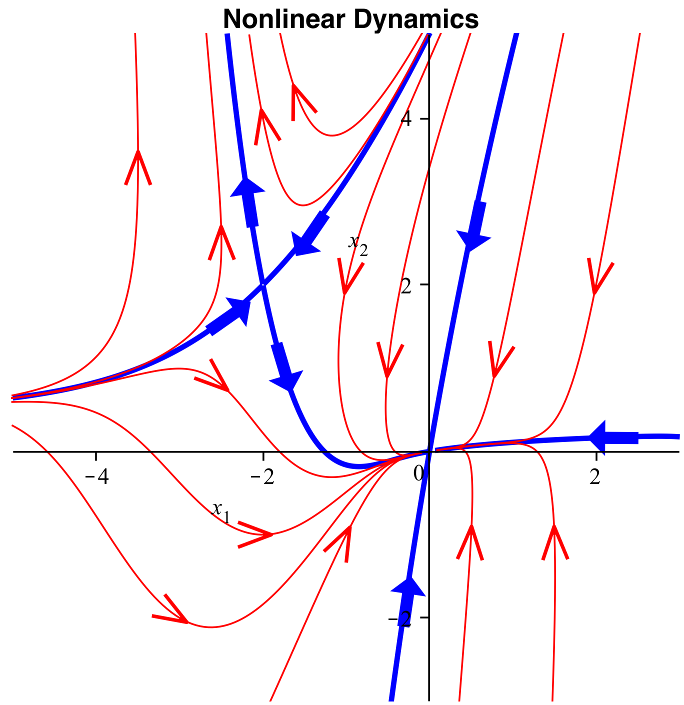
Linear Approx Dynamics Near (0,0)



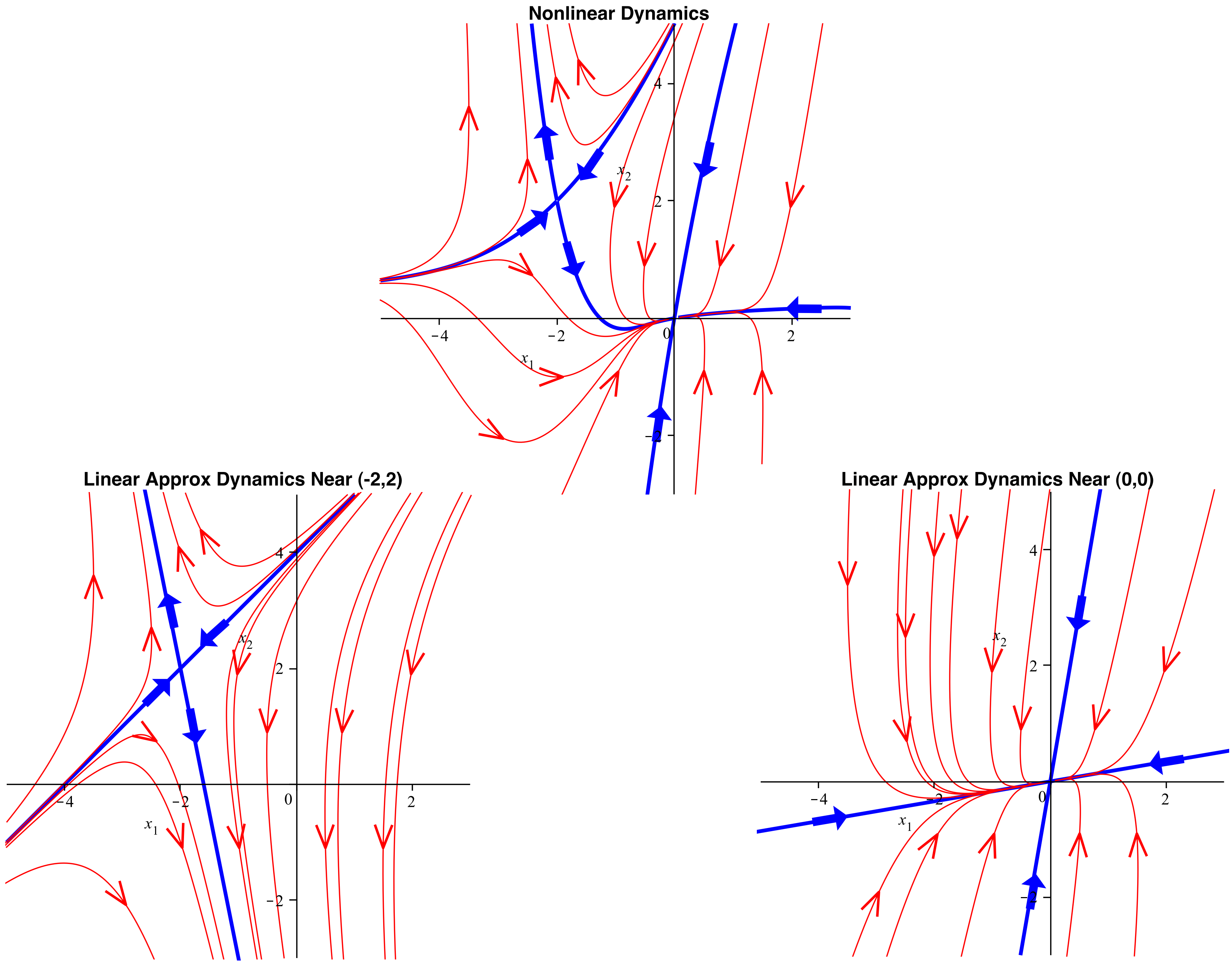


Nonlinear Dynamics Near Equilibria

Example 5. Global phase portrait of the nonlinear system



Example 5. Global phase portrait of the nonlinear system



Example 6 (Neutral Eigenvalue)

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find all equilibria.
- For each equilibrium, give the linear approximating system near it.
- Sketch the phase portrait of the linear approximating system.
- Determine whether each equilibrium is stable or unstable with respect to the nonlinear system.

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 = 0 \\ (2) \quad -x_2 + x_1^2 = 0 \end{cases}$$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 = 0 \\ (2) \quad -x_2 + x_1^2 = 0 \end{cases}$$

From (2), $x_2 = x_1^2$.

Substitute this in (1):

$$x_1^3 + x_1^4 + x_1^5 = 0 \Rightarrow x_1^3(1 + x_1 + x_1^2) = 0 \Rightarrow x_1 = 0$$

Example 6 (continued). Find equilibria.

$$\begin{cases} x_1' = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ x_2' = -x_2 + x_1^2 \end{cases}$$

- Find equilibria, by solving $\vec{\mathbf{f}}(\vec{\mathbf{x}}) = 0$, that is, $\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$

$$\begin{cases} (1) \quad 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 = 0 \\ (2) \quad \;\; - x_2 + x_1^2 = 0 \end{cases}$$

From (2), $x_2 = x_1^2$.

Substitute this in (1):

$$x_1^3 + x_1^4 + x_1^5 = 0 \Rightarrow x_1^3(1 + x_1 + x_1^2) = 0 \Rightarrow x_1 = 0$$

From $x_2 = x_1^2$ it follows $x_2 = 0$.

$\Rightarrow$ **Only one equilibrium:** $(x_1, x_2) = (0, 0)$.

Example 6 (continued).

Linear approx sys near the equilibrium $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

Example 6 (continued).

Linear approx sys near the equilibrium $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_1^2 + 5x_1^4 & 2x_1 + 2x_2 \\ 2x_1 & -1 \end{bmatrix}$$

Example 6 (continued).

Linear approx sys near the equilibrium $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_1^2 + 5x_1^4 & 2x_1 + 2x_2 \\ 2x_1 & -1 \end{bmatrix}$$

- Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

Example 6 (continued).

Linear approx sys near the equilibrium $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_1^2 + 5x_1^4 & 2x_1 + 2x_2 \\ 2x_1 & -1 \end{bmatrix}$$

- Evaluate J at equilibrium $(x_1,x_2) = (0,0)$: $J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$
- The linear approximating system near $(0,0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 6. (continued)

Linear Approximate Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$: $\quad \vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$

Example 6. (continued)

Linear Approximate Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:

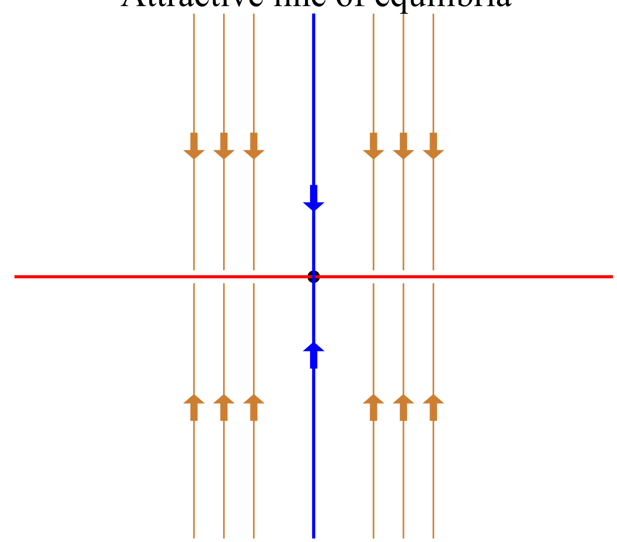
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

Example 6. (continued)

Linear Approximate Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
- Thus, the linear approximate dynamics has an attractive line of equilibria.



Example 6. (continued)

Linear Approximate Dynamics near $(0,0)$.

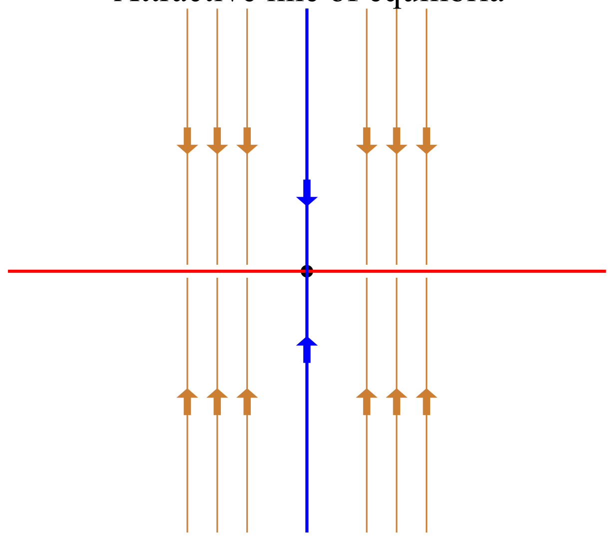
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1,x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:

$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

- Thus, the linear approximate dynamics has an attractive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0,0)$.

Linear approximating dynamics:
Attractive line of equilibria

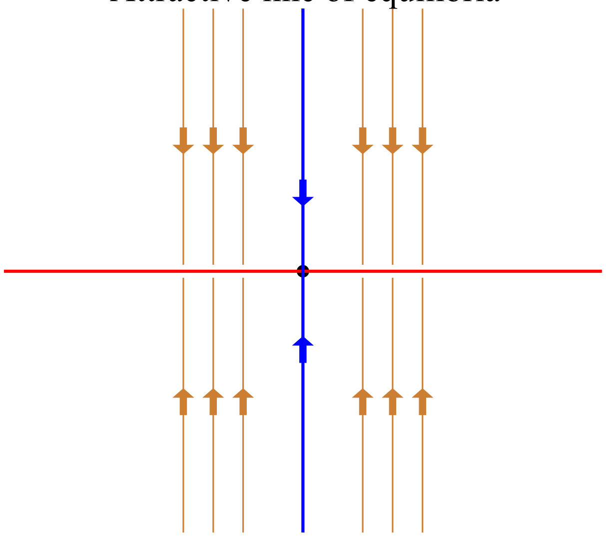


Example 6. (continued)

Linear Approximate Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ f_2(x_1, x_2) = -x_2 + x_1^2 \end{cases}$$

- Linear approx system near the equilibrium$(0, 0)$: $\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = -1 < 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
- Thus, the linear approximate dynamics has an attractive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0, 0)$.
- In other words, the linear analysis fails to determine the local nonlinear dynamics near $(0, 0)$.

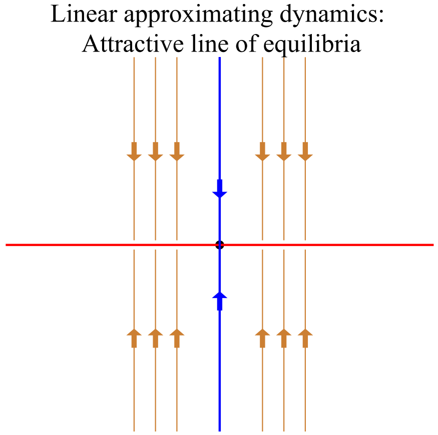


Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$

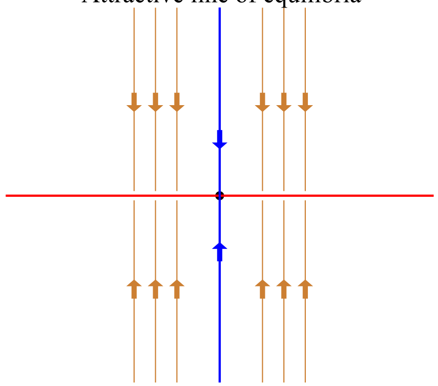


Example 6. (continued)

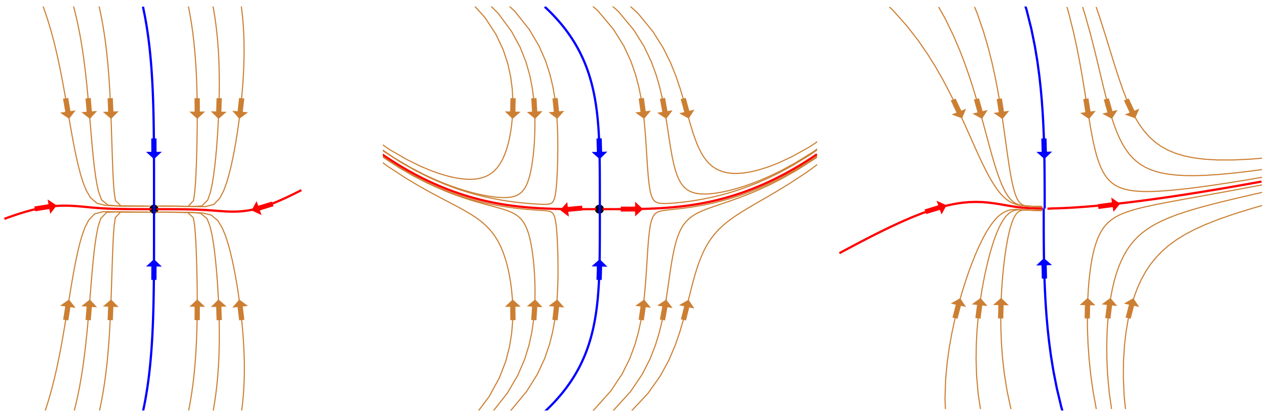
Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$



The following is an *incomplete* list of the possible local phase portraits of the nonlinear system near $(0, 0)$:



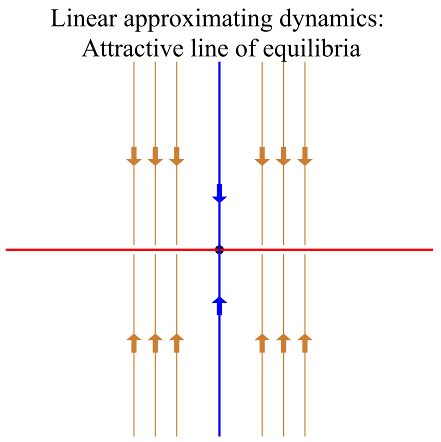
et cetera

Example 6. (continued)

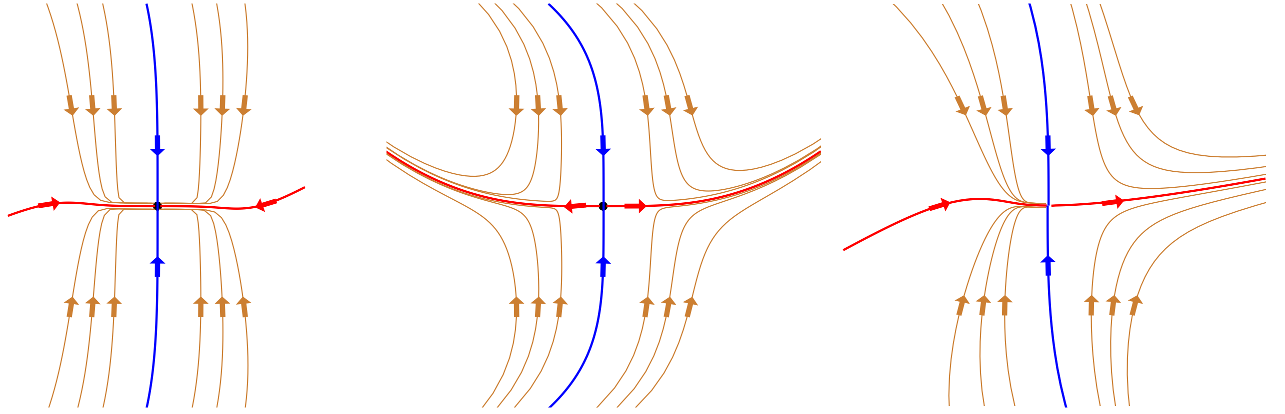
Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$



The following is an *incomplete* list of the possible local phase portraits of the nonlinear system near $(0, 0)$:



et cetera

To determine the correct picture, need advanced nonlinear theories: normal forms, center manifolds, $\cdots$.

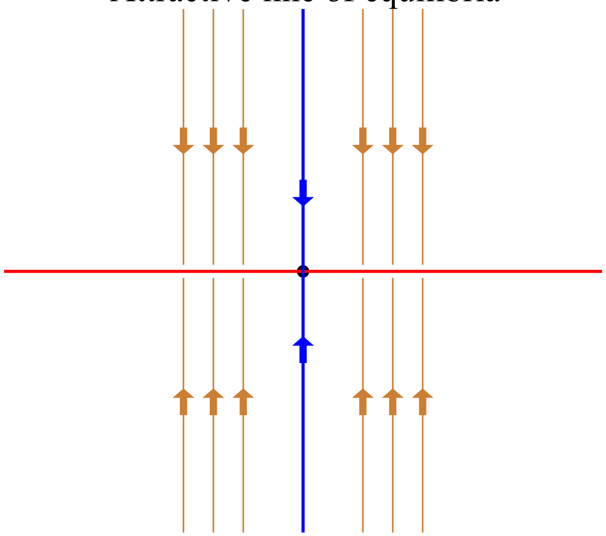
Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$

Linear approximating dynamics: Attractive line of equilibria

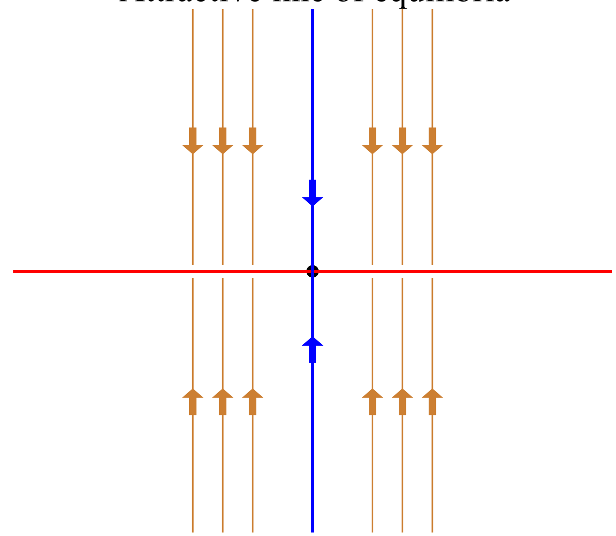


Example 6. (continued)

Linear approx system for $(x_1, x_2) \approx (0, 0)$:

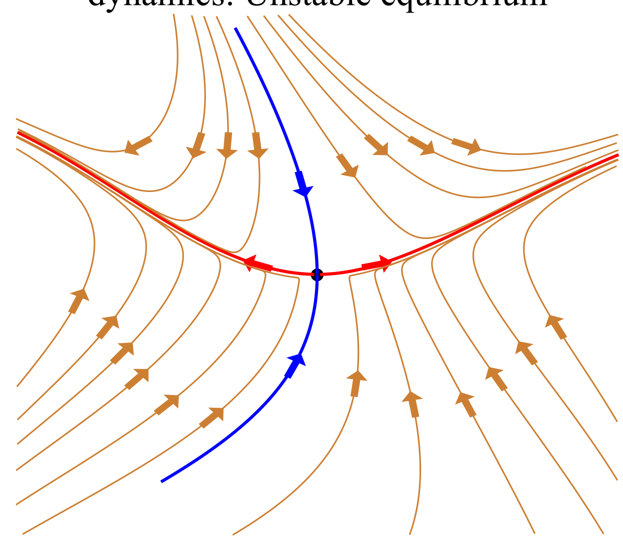
$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$

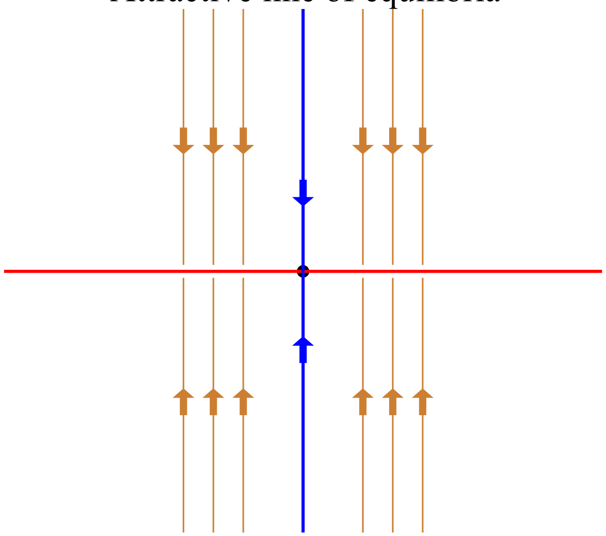


The actual local phase portrait of the nonlinear system near $(0, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix}$$



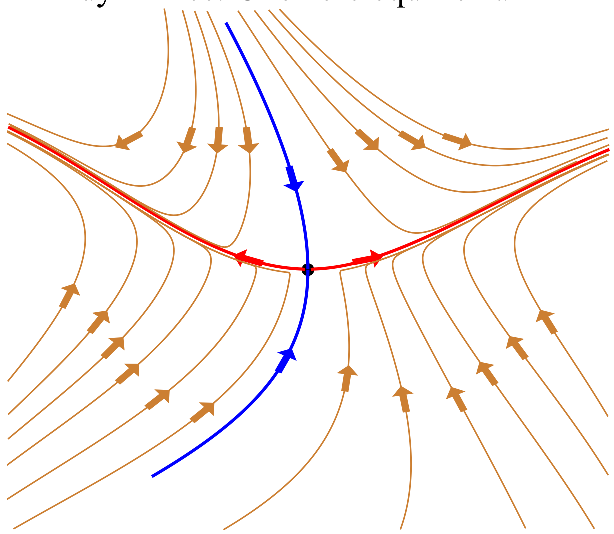
Example 6. (continued)



Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases}$$



The actual local phase portrait of the nonlinear system near $(0,0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 - x_1^3 + x_1^5 \\ -x_2 + x_1^2 \end{bmatrix}$$

- Impossible to get this by the linear approximation alone.
- Advanced *nonlinear* tools (center manifolds, ...) can get us this picture.

Example 7 (Neutral Eigenvalue)

$$\begin{cases} x_1' = x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2' = x_2 + \frac{1}{2} x_1^3 \end{cases}$$

- Give the linear approximating system near the equilibrium $(0, 0)$. Sketch the phase portrait of the linear approx system.
- Determine whether $(0, 0)$ is stable or unstable with respect to the nonlinear system.
- Sketch the local phase portrait of the nonlinear system near $(0, 0)$

Example 7 (a) Linear approx system near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

Example 7 (a) Linear approx system near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2 + \frac{1}{2} x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2} x_1^3 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + 4x_1^3 & x_1 + \frac{3}{5} x_2^2 \\ \frac{3}{2} x_1^2 & 1 \end{bmatrix}$$

Example 7 (a) Linear approx system near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2 + \frac{1}{2} x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2} x_1^3 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + 4x_1^3 & x_1 + \frac{3}{5} x_2^2 \\ \frac{3}{2} x_1^2 & 1 \end{bmatrix}$$

- Evaluate J at equilibrium $(x_1, x_2) = (0, 0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example 7 (a) Linear approx system near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

- Calculate the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + 4x_1^3 & x_1 + \frac{3}{5}x_2^2 \\ \frac{3}{2}x_1^2 & 1 \end{bmatrix}$$

- Evaluate J at equilibrium $(x_1, x_2) = (0,0)$: $\quad J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- The linear approximating system near $(0,0)$ is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 7 (a) Linear Approx Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1,x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$:
  $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 7 (a) Linear Approx Dynamics near $(0, 0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1, x_2) = x_1 x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1, x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

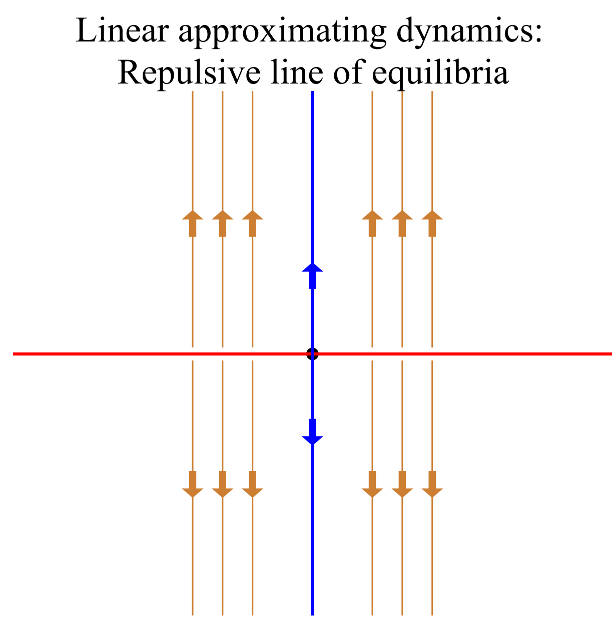
- Linear approx system near the equilibrium$(0, 0)$:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = 1 > 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

Example 7 (a) Linear Approx Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1,x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

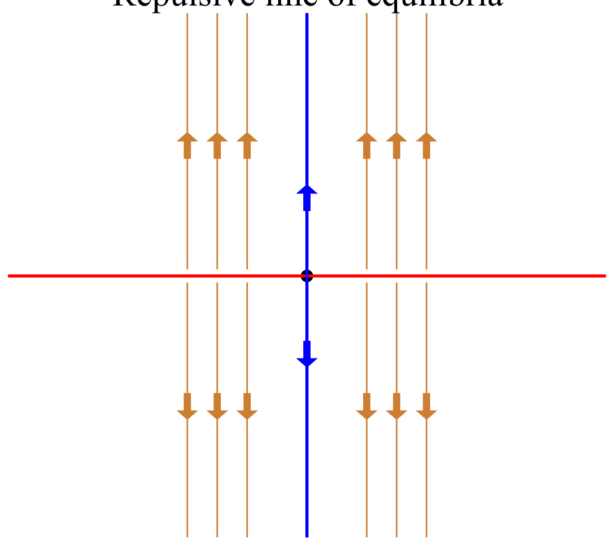
- Linear approx system near the equilibrium$(0,0)$:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = 1 > 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
- Thus, the linear approximate dynamics has a repulsive line of equilibria.



Example 7 (a) Linear Approx Dynamics near $(0,0)$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ x_2 + \frac{1}{2}x_1^3 \end{bmatrix} \qquad \begin{cases} f_1(x_1,x_2) = x_1x_2 + \frac{1}{5}x_2^3 + x_1^4 \\ f_2(x_1,x_2) = x_2 + \frac{1}{2}x_1^3 \end{cases}$$

- Linear approx system near the equilibrium$(0,0)$:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
- Eigenvalues & eigenvectors:
$$\begin{cases} \lambda_1 = 0, & \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \lambda_2 = 1 > 0, & \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
- Thus, the linear approximate dynamics has a repulsive line of equilibria.
- Since there is a neutral eigenvalue $\lambda_1 = 0$, it is possible that the nonlinear dynamics is non-equivalent to the linear dynamics near $(0,0)$.

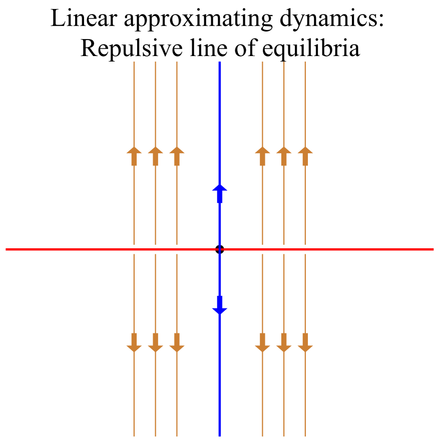


Example 7. (b)(c) Local nonlinear dynamics near $(0, 0)$

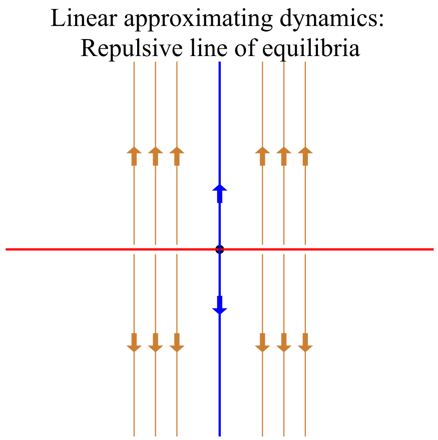
Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$



Example 7. (b)(c) Local nonlinear dynamics near $(0,0)$

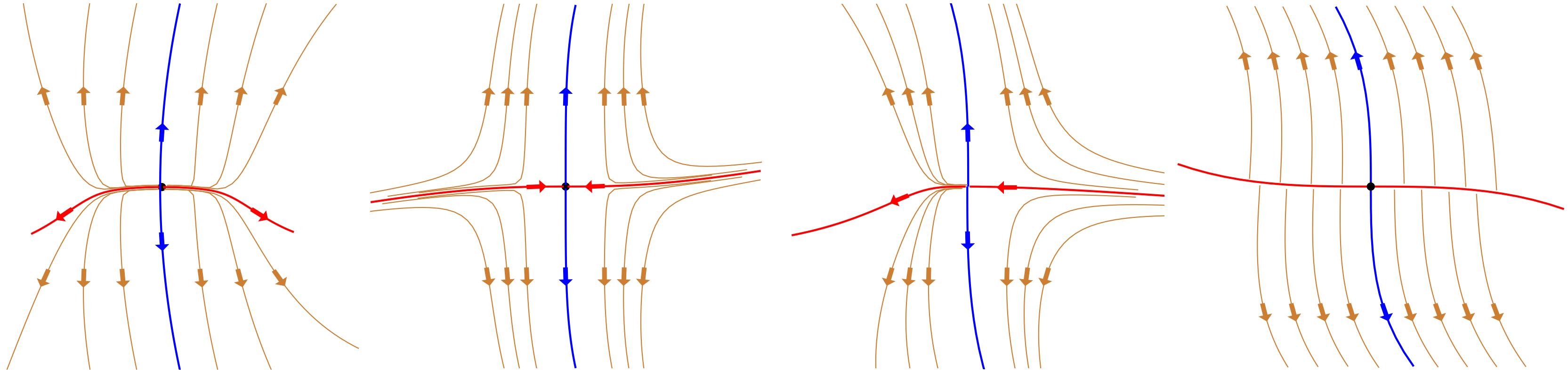


Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$

An *incomplete* list of possible nonlinear dynamics near $(0,0)$:

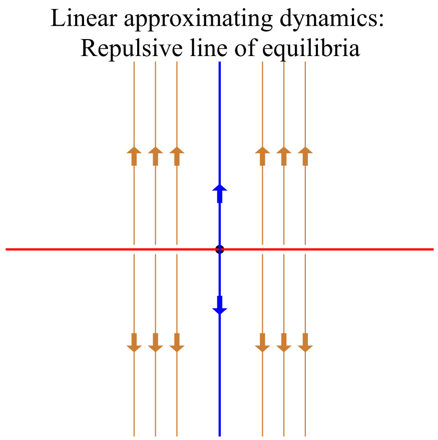


Example 7. (b)(c) Local nonlinear dynamics near $(0, 0)$

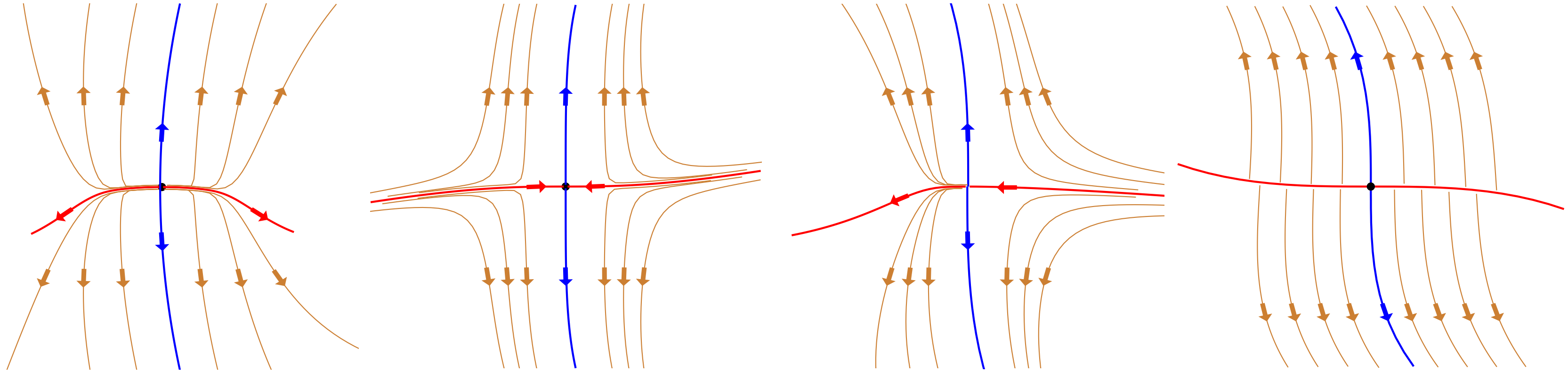
Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$

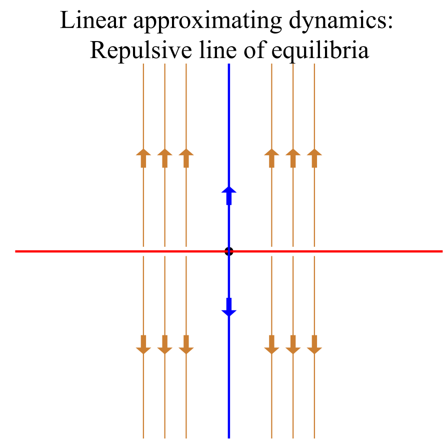


An *incomplete* list of possible nonlinear dynamics near $(0, 0)$:



- Linear analysis alone cannot determine the correct picture.

Example 7. (b)(c) Local nonlinear dynamics near $(0,0)$

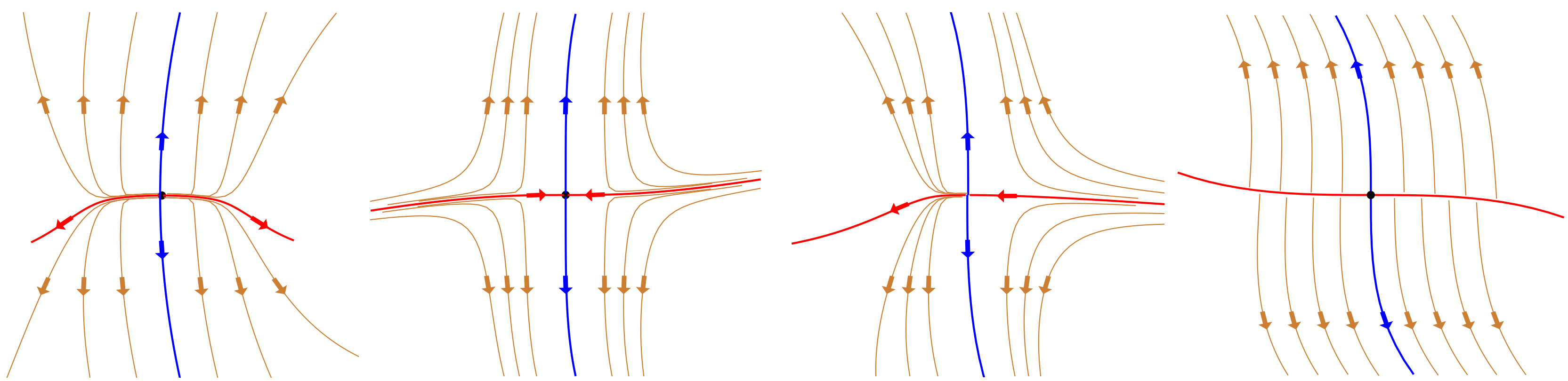


Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$

An *incomplete* list of possible nonlinear dynamics near $(0,0)$:



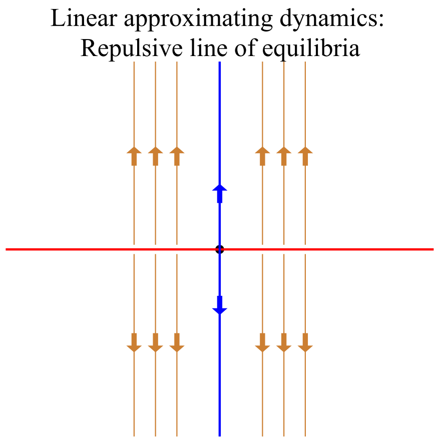
- Linear analysis alone cannot determine the correct picture.
- But we do know $(0,0)$ is unstable in the nonlinear system.

Example 7. (b)(c) Local nonlinear dynamics near $(0,0)$

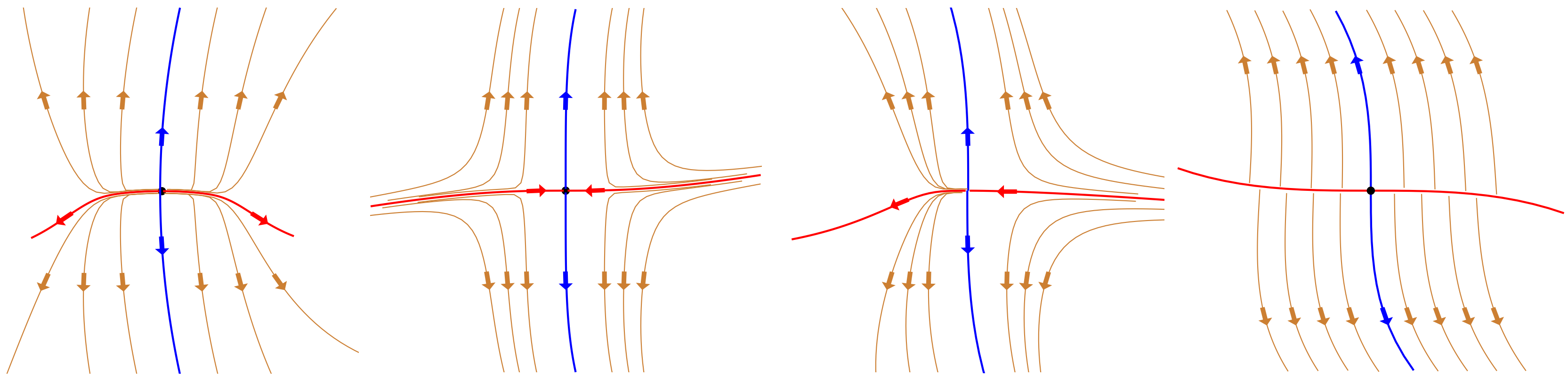
Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

Eigenvalues $\begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$



An *incomplete* list of possible nonlinear dynamics near $(0,0)$:



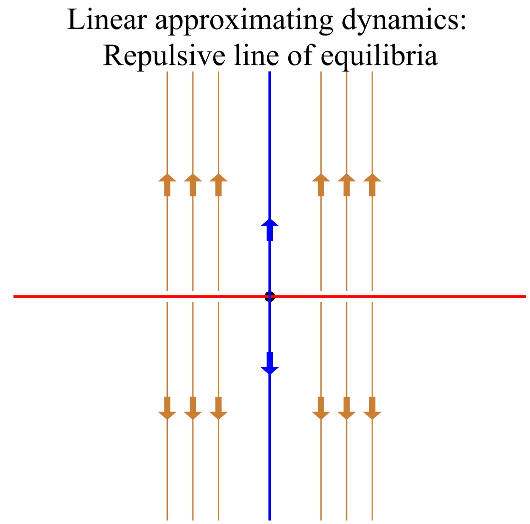
- Linear analysis alone cannot determine the correct picture.
- But we do know $(0,0)$ is unstable in the nonlinear system.
- **Reason:** since $\lambda_2 = 1 > 0$, solutions along this eigenspace will grow, with the growth rate ≈ 1, even in the nonlinear system.

Example 7. Summary.

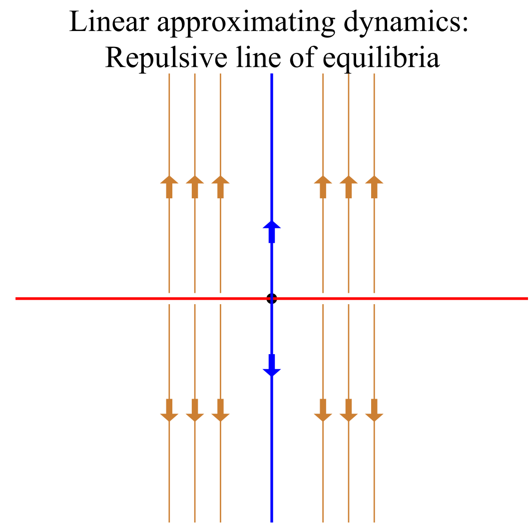
Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$



Example 7. Summary.



Linear approx system for $(x_1, x_2) \approx (0, 0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$

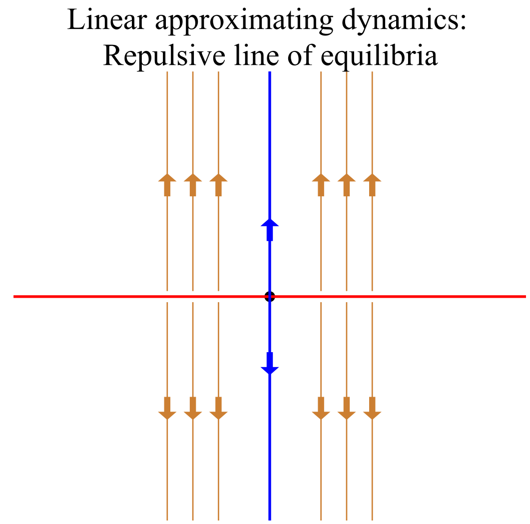
- Since one of the eigenvalues is > 0,
  the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0, 0)$.

Example 7. Summary.

Linear approx system for $(x_1, x_2) \approx (0,0)$:

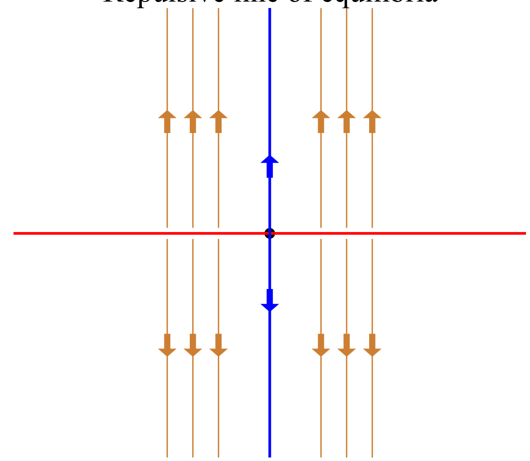
$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$



- Since one of the eigenvalues is > 0, the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0,0)$.
- Since one of the eigenvalues is $= 0$ (neutral), the linear aprroximation $\not\Rightarrow$ the nonlinear local phase portrait.

Example 7. Summary.

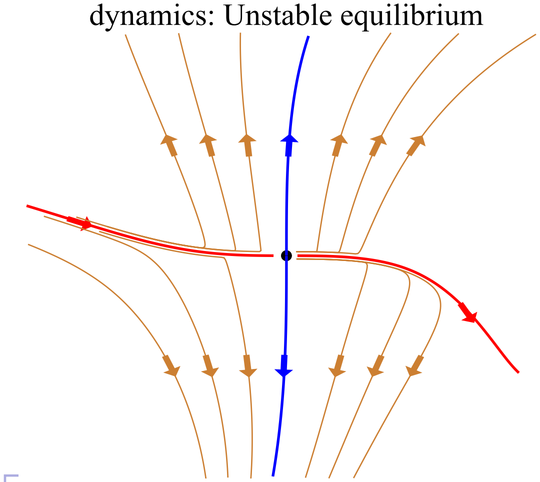


Linear approx system for $(x_1, x_2) \approx (0,0)$:

$$\vec{\mathbf{x}}\,' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$$

$$\text{Eigenvalues} \begin{cases} \lambda_1 = 0 & \text{(neutral)} \\ \lambda_2 = 1 > 0 & \text{(instability)} \end{cases}$$

- Since one of the eigenvalues is > 0, the linear aprroximation $\Rightarrow$ the nonlinear instability of $(0,0)$.
- Since one of the eigenvalues is $= 0$ (neutral), the linear aprroximation $\not\Rightarrow$ the nonlinear local phase portrait.



- Advanced *nonlinear* tools (center manifolds, ...) can give the local phase portrait of the nonlinear system near $(0,0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \frac{1}{5} x_2^3 + x_1^4 \\ x_2 + \frac{1}{2} x_1^3 \end{bmatrix}$$