First Order Autonomous Equations

— Linear Approximating Equations Near Equilibria

Xu-Yan Chen

Objective: Assume that a diff eq $y' = f(y)$ has an equilibrium $y = b$. (In other words, $f(b) = 0$.)

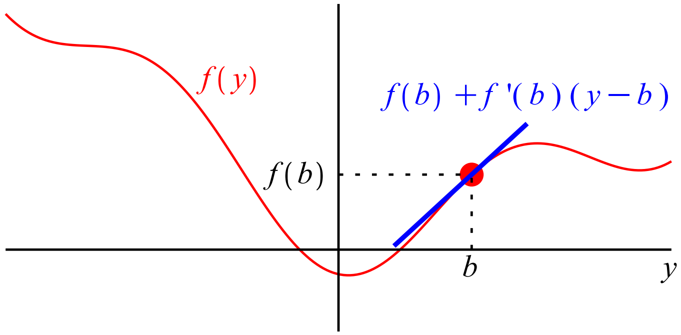
We will see that the derivative $f'(b)$ can help to give

- Local Phase Portrait near $y \approx b$;
- Stability/Instability of Equilibrium $y = b$;
- Linear Approximating Diff Eqs near $y \approx b$.

Review of Calculus

Derivative $f'(b) \Rightarrow$ Slope $\Rightarrow$ Tangent
$\Rightarrow$ Linear approximation near b:

$$f(y) \approx f(b) + f'(b)(y-b) \quad \text{for } y \text{ near } b.$$



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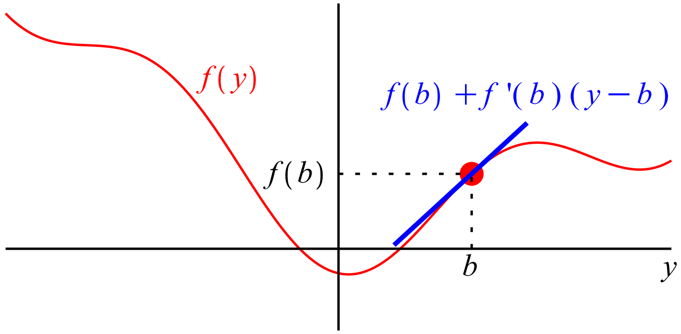

Example: $f(y) = y - y^3$ has derivative $f'(y) = 1 - 3y^2$.

At $y = \frac{1}{3}$: $f(\frac{1}{3}) = \frac{8}{27}, f'(\frac{1}{3}) = \frac{2}{3} \Rightarrow$

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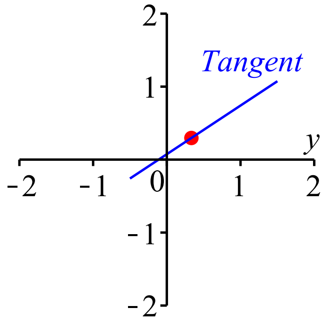

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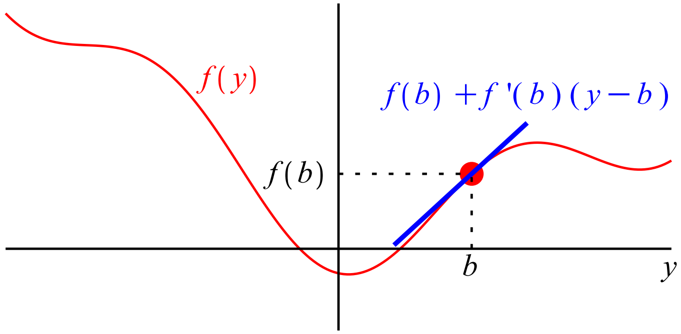

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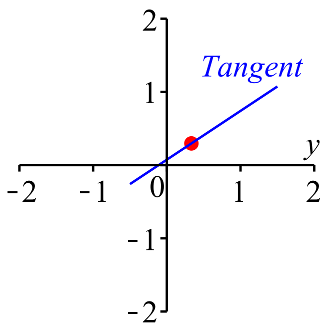

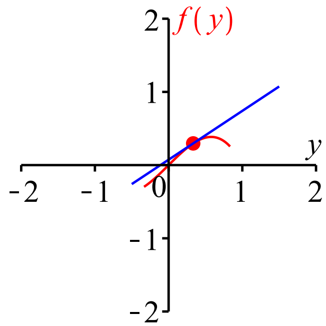

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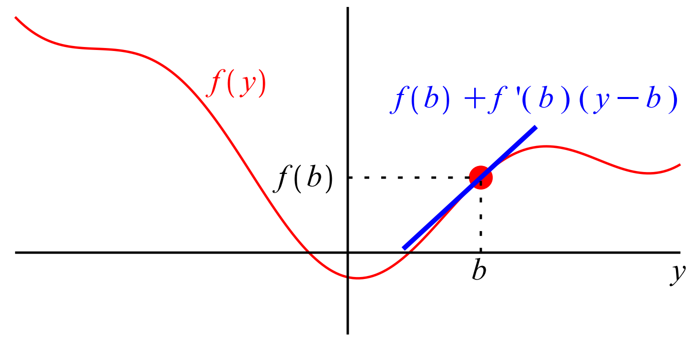

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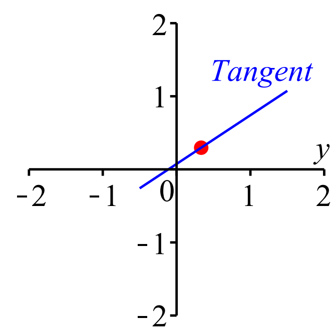

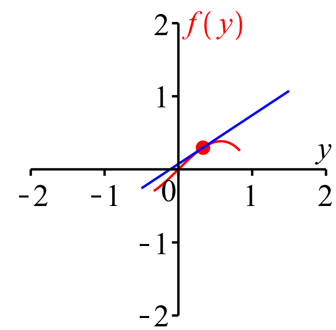

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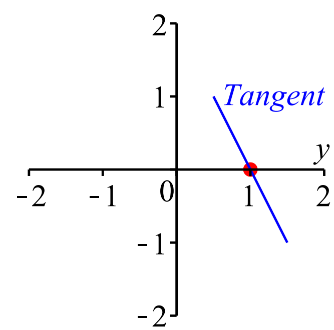

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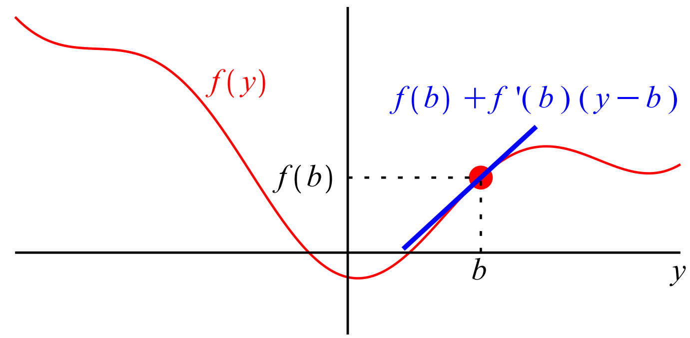

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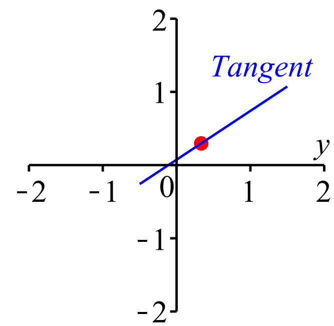

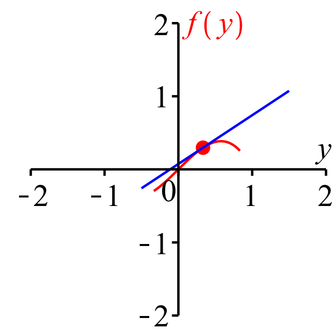

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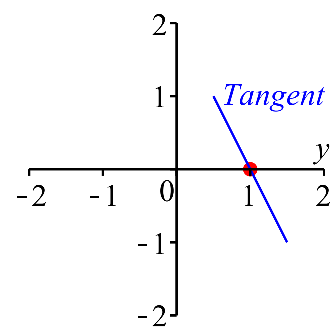

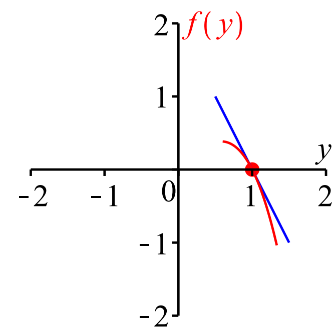

Nonlinear Diff Eq

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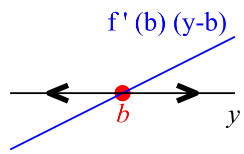

Unstable (repulsive)

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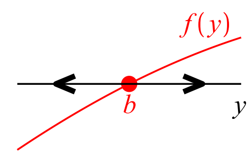
Nonlin Phase Portrait Near b



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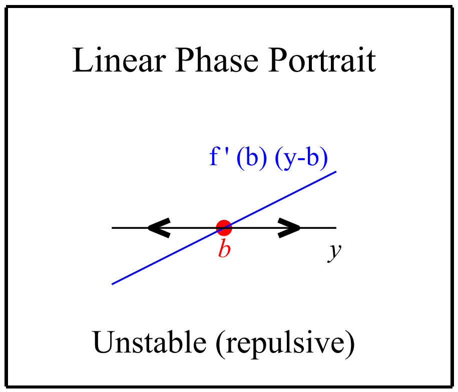

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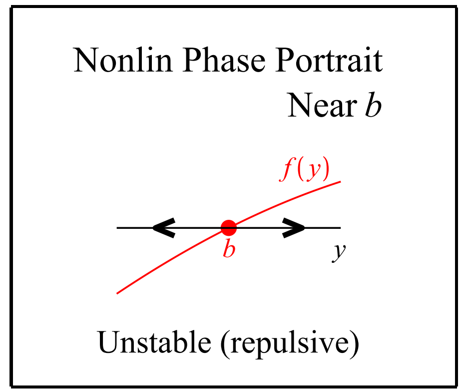

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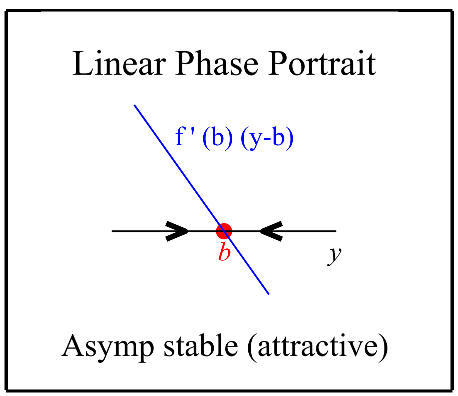

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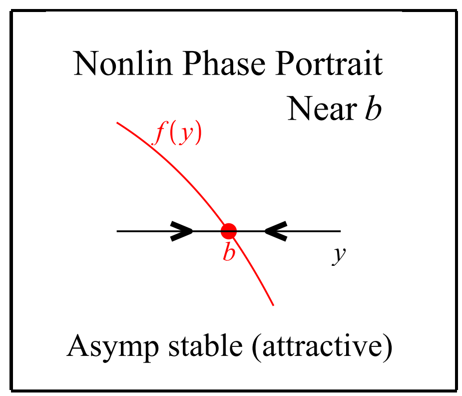

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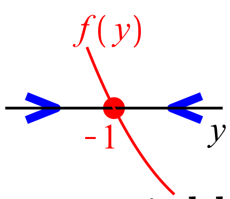

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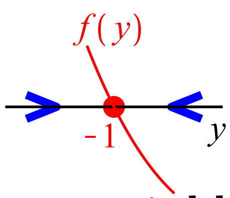

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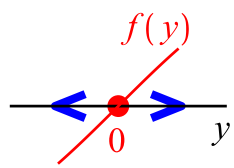

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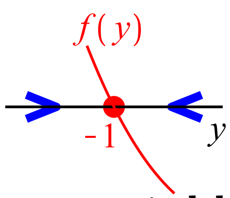

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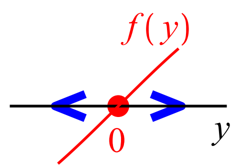

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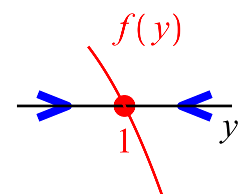
Local Phase Portrait for y near 1



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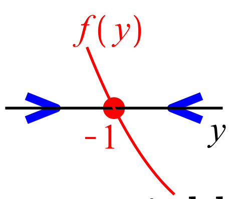

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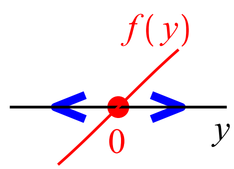

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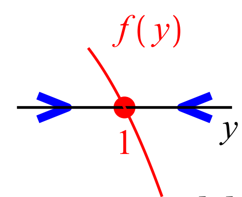

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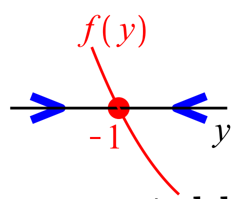

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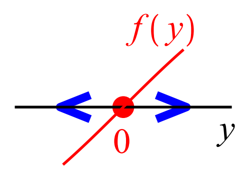

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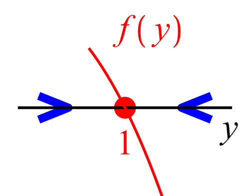

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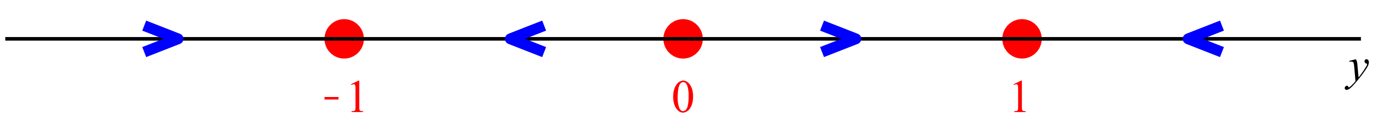
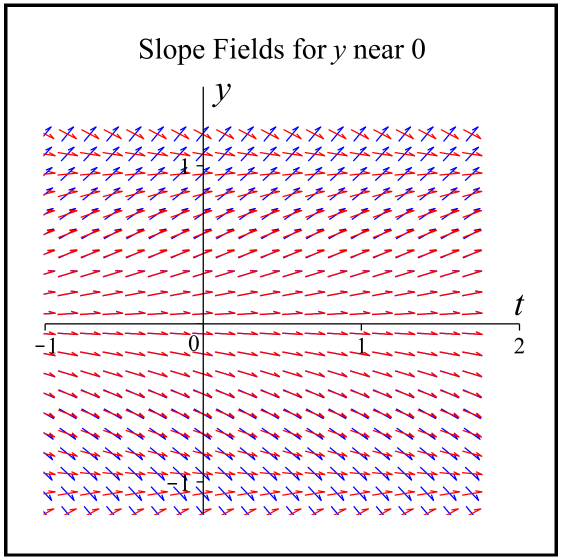
Global Phase Portrait

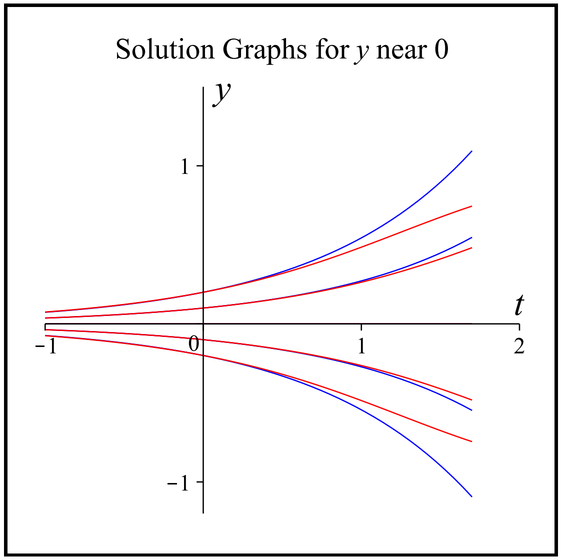


A Comparison of the Nonlinear Eq & Linear Approx Eqs

For y near 0

$$\begin{cases} \text{Nonlinear Eq:} & y' = y - y^3 \\ \text{Lin Approx Eq:} & y' = y \end{cases}$$





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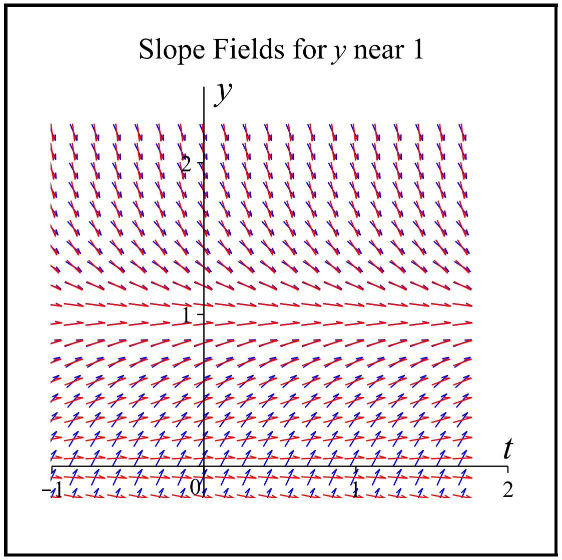

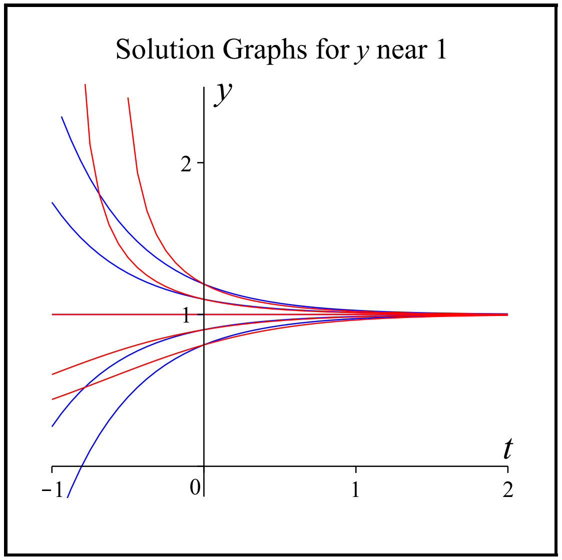

Example 2: $y' = 2 + 3y - 2e^{4y}$.

- Verify that $y = 0$ is an equilibrium.
- Give linear approximating equation for y near 0.
- Determine whether $y = 0$ is stable or unstable.
- Sketch the local phase portrait near $y \approx 0$.

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$$\left.\begin{array}{l} f(0) = 0 \\ f'(0) = -5 \end{array}\right\} \Rightarrow \left[\begin{array}{c} \textbf{Lin Approx Eq} \\ \textbf{for } y \textbf{ near } 0 \\ y' = -5y \end{array}\right]$$

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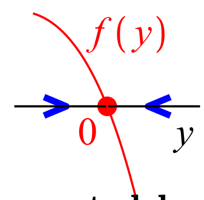
Nonlin Local Phase Portrait for y near 0



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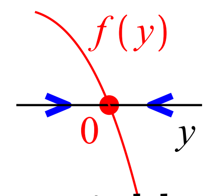

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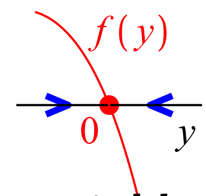

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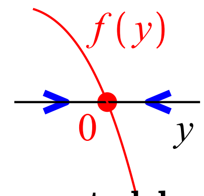

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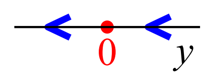

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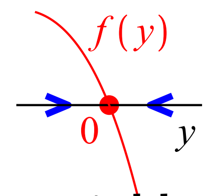

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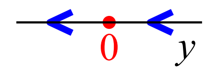

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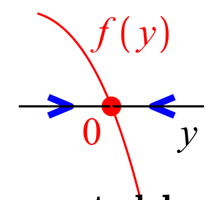

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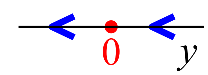

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There is another equilibrium $b = -0.608\cdots$

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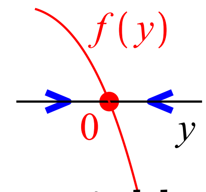

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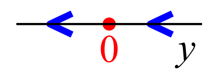

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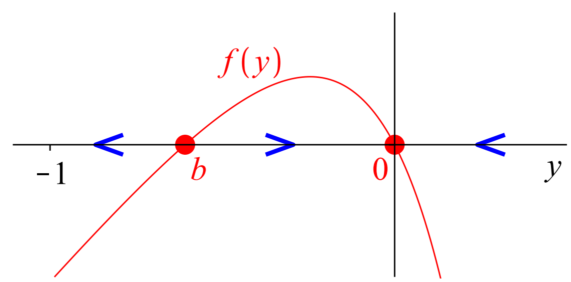
The True Global Phase Portrait



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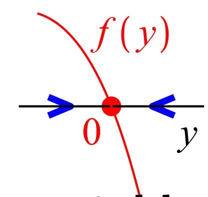

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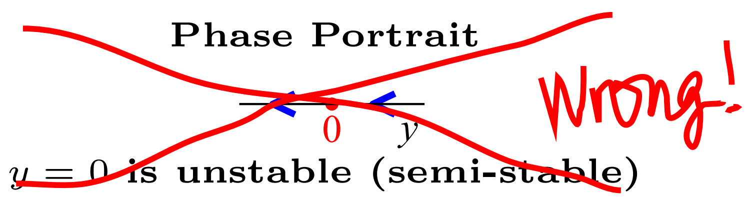

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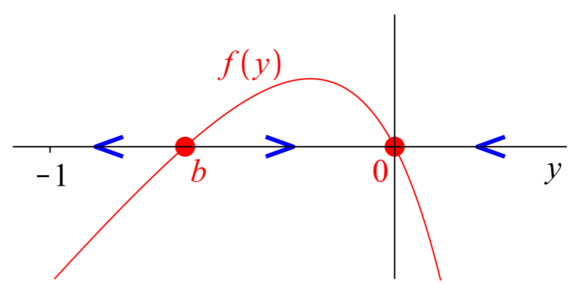

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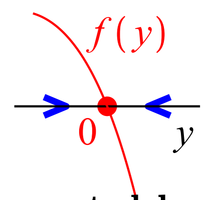

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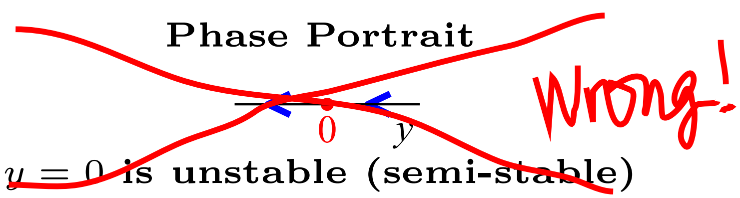

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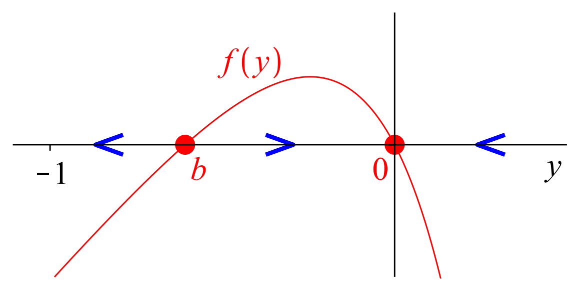
Lesson: The linear approximation gives *local* dynamics only.

Lesson: For *global* dynamics, need to analyse *all* equilbria.

The True Global Phase Portrait



Example 3: **Study** $y' = (y+1)^3(y-1)(y-2)^2$.

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$f'(1) = 8$

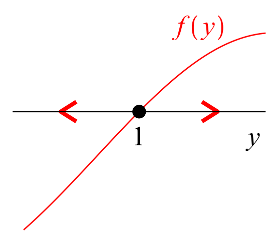
$\Downarrow$

Lin Approx Eq

$y' = 8(y-1)$

$\Downarrow$

Local Phase Portrait
for y near 1



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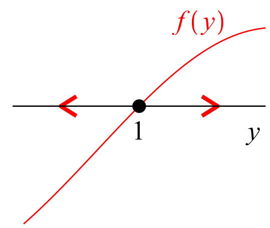
$f'(-1) = 0$	$f'(1) = 8$
$\Downarrow$	$\Downarrow$
Lin Approx Eq	**Lin Approx Eq**
$y' = 0$	$y' = 8(y-1)$
	$\Downarrow$
	Local Phase Portrait for y near 1
	
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Local Phase Portrait
for y near -1

?

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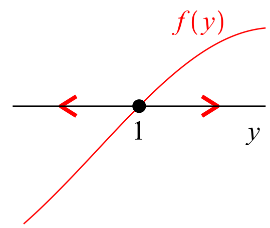

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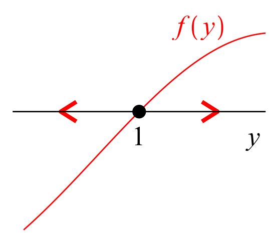

unstable (repulsive)

$f'(2) = 0$

$\Downarrow$

Lin Approx Eq

$y' = 0$

$\Downarrow$

Local Phase Portrait for y near 2

?

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$\Downarrow$

Lin Approx Eq

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Local Phase Portrait for y near -1

?

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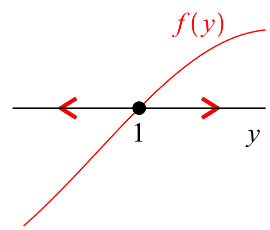
$\Downarrow$

Local Phase Portrait for y near 1



unstable (repulsive)

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Lin Approx Eq

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Local Phase Portrait for y near 2

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Near the equilibria where $f'(b) = 0$,
the linear analysis fails to determine the local nonlinear dynamics.

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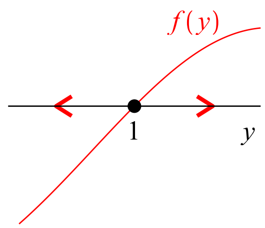
Local Phase Portrait for y near 1



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Local Phase Portrait for y near 2

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Near the equilibria where $f'(b) = 0$,
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Alternative Approach:

Study the signs of $y' = f(y)$ at locations away from the equilibria.

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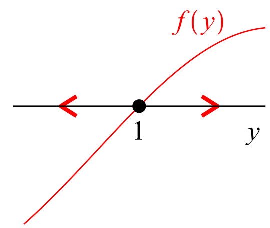
$\Downarrow$	$\Downarrow$	$\Downarrow$
Lin Approx Eq	**Lin Approx Eq**	**Lin Approx Eq**
$y' = 0$	$y' = 8(y-1)$	$y' = 0$
$\Downarrow$	$\Downarrow$	$\Downarrow$
Local Phase Portrait for y near -1	**Local Phase Portrait for y near 1**	**Local Phase Portrait for y near 2**
?		?
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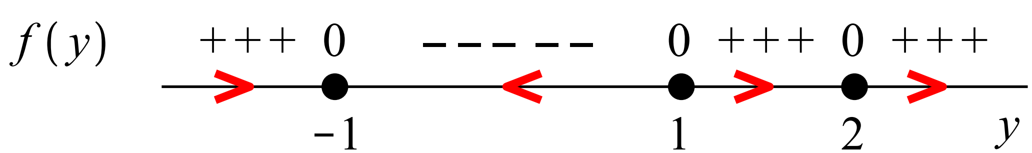
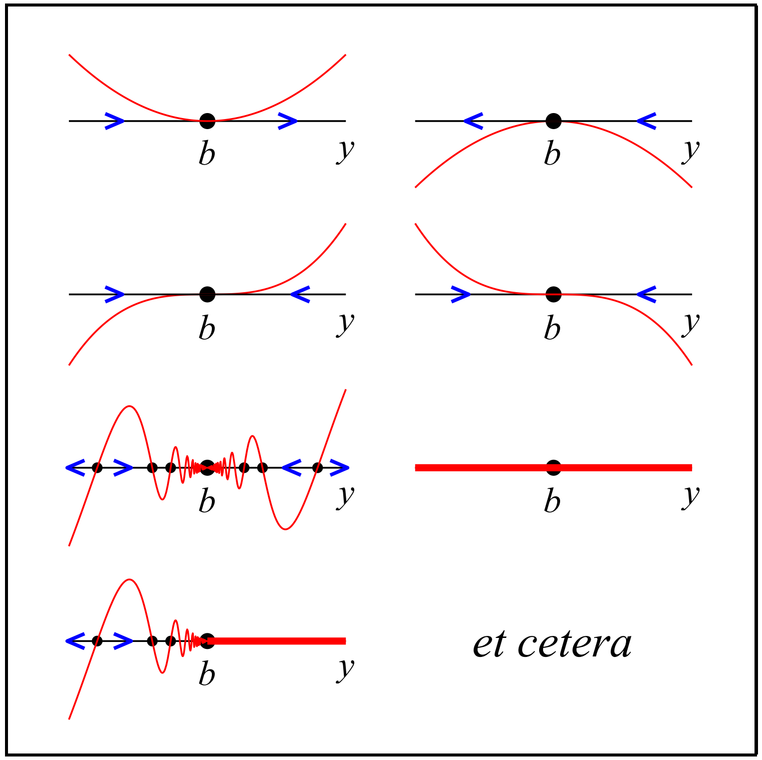
$\Downarrow$



Lesson: When $f(b) = 0$ and $f'(b) = 0$, the linear approximation alone is insufficient in determining the nonlinear local phase portrait near $y \approx b$.

The following is an *incomplete* list
of the possible local pictures when $f(b) = 0$ and $f'(b) = 0$:



To determine the correct picture, $\left\{\begin{array}{l}\text{use higher degree approximations,}\\ \text{study the signs of } f(y),\\ \text{or use other more advanced methods.}\end{array}\right.$