Solution Structure of Systems of Diff Eqs

2-D Homog. Lin. Systems of Diff Eq's

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 \\ \dfrac{dx_2}{dt} = a_{21}(t)\,x_1 + a_{22}(t)\,x_2 \end{cases}$$

Matrix - Vector Form

$$\frac{d\vec{x}}{dt} = A(t)\,\vec{x} \qquad \text{where} \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$$

Gen. Sol. Structure

$$\vec{x}(t) = C_1\,\vec{w}_1(t) + C_2\,\vec{w}_2(t)$$

where C_1, C_2 : free parameters

$\vec{w}_1(t), \vec{w}_2(t)$: lin. independent sol's of the Homog. System

Sol. Method

Does not exist in general

2-D Nonhomog Lin. Systems of Diff Eq's

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + b_1(t) \\ \dfrac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + b_2(t) \end{cases}$$

Matrix - Vector Form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{b}(t)$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$

$\vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$

Gen Sol. Structure

$$\vec{x}(t) = \vec{x}_p(t) + \underbrace{c_1\vec{w}_1(t) + c_2\vec{w}_2(t)}$$

where $\vec{x}_p(t)$: a particular sol of Nonhomog Sys.

$c_1\vec{w}_1(t) + c_2\vec{w}_2(t)$: "Complementary sol's."

Solution Method

Bad News • No Sol. Method for getting "Complementary sol's".

Good News • If "Complementary sol's" have already been obtained, the method of "variation of parameters" can give us $\vec{x}_p(t)$.

$\boxed{\text{2-D Nonlinear Systems of Diff Eq's}}$

Vector Form:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, x_2) \\[2ex] \dfrac{dx_2}{dt} = f_2(t, x_1, x_2) \end{cases} \qquad \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x})$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\vec{f}(t, \vec{x}) = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2) \end{bmatrix}$

Gen. Sol's Structure

Gen. Sol's Contain Two free parameters C_1, C_2.

Sol. Method

No general method exists

$\boxed{\text{3-D Homog. Lin. Systems of Diff Eq's}}$

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + a_{13}(t)\,x_3 \\ \dfrac{dx_2}{dt} = a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + a_{23}(t)\,x_3 \\ \dfrac{dx_3}{dt} = a_{31}(t)\,x_1 + a_{32}(t)\,x_2 + a_{33}(t)\,x_3 \end{cases}$$

Matrix-Vector Form

$$\frac{d\vec{x}}{dt} = A(t)\,\vec{x}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

3×3 matrix

Gen. Sol. Structure

$$\vec{x}(t) = C_1\,\vec{w}_1(t) + C_2\,\vec{w}_2(t) + C_3\,\vec{w}_3(t)$$

where C_1, C_2, C_3 are free parameters

$\vec{w}_1(t), \vec{w}_2(t), \vec{w}_3(t)$ are lin. independent sol's of the Homog. System

Sol. Method

Does NOT Exist in general.

n-D Homog. Lin. Systems of Diff Eq's

$$\begin{cases} \frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ \frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{cases}$$

Matrix - Vector Form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$n \times n$ matrix

Gen. Sol. Structure

$$\vec{x}(t) = c_1\vec{w}_1(t) + c_2\vec{w}_2(t) + \cdots + c_n\vec{w}_n(t)$$

where $c_1, c_2, \cdots, c_n$ are free parameters,

$\vec{w}_1(t), \vec{w}_2(t), \cdots, \vec{w}_n(t)$ are lin. indep. sol's of the Homog. System

Sol. Method

Does NOT exist in general.

n-D Homog. Lin. Systems of Diff Eq's with **Constant Coefficients**

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

Matrix-Vector Form $\dfrac{d\vec{x}}{dt} = A\vec{x}$,

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$: $n \times n$ Constant matrix.

Gen. Sol. Structure $\vec{x}(t) = C_1\vec{w}_1(t) + \cdots + C_n\vec{w}_n(t)$,

where $C_1, \cdots, C_n$ are free parameters,
$\vec{w}_1(t), \cdots, \vec{w}_n(t)$ are linearly indep. sol's of $\dfrac{d\vec{x}}{dt} = A\vec{x}$.

The Most Important Method of Constructing Particular Solutions

If $A\vec{u} = \lambda\vec{u}$ with λ being a scalar & $\vec{u}$ being a nonzero vector

(in other words, if λ is an eigenvalue of A with a corresponding eigenvector $\vec{u}$)

then $e^{\lambda t}\vec{u}$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$

Example Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \vec{x}$

Solution

Example Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \vec{x}$

$\boxed{\text{Solution}}$ Eigenvalues $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (1)(2) = \lambda^2 - \lambda - 2,$
$= (\lambda+1)(\lambda-2).$

$\lambda_1 = -1, \quad \lambda_2 = 2.$

Eigenvectors for $\lambda_1 = -1$ $\quad (A - \lambda_1 I)\vec{x} = \vec{0}.$

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$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \Leftrightarrow 2x_1 + x_2 = 0 \Leftrightarrow x_1 = -\frac{1}{2}x_2 . \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$

Eigenvectors for $\lambda_2 = 2$ $\quad (A-\lambda_2 I)\vec{x} = \vec{0}.$

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Eigenvectors for $\lambda_2 = 2$ $\quad (A-\lambda_2 I)\vec{x} = \vec{0}. \quad (A-2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -x_1 + x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \Leftrightarrow x_1 = x_2 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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General Solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$: $\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

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General Solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$: $\quad \vec{x}(t) = c_1 e^{-t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\boxed{\text{Remark}}$

Alternative Expressions : $\quad \vec{x}(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

$\vdots$

There are infinitely many possible expressions.

Ex. Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

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- Init. Cond. $\vec{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

  or, equivalently, $\begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

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- Row Reduce

$$\left[\begin{array}{cc:c} -\frac{1}{2} & 1 & 3 \\ 1 & 1 & 5 \end{array}\right] \underset{R1 \leftrightarrow R2}{\sim} \left[\begin{array}{cc:c} 1 & 1 & 5 \\ -\frac{1}{2} & 1 & 3 \end{array}\right] \underset{2R2}{\sim} \left[\begin{array}{cc:c} 1 & 1 & 5 \\ -1 & 2 & 6 \end{array}\right] \; R2+R1$$

$$\sim \left[\begin{array}{cc:c} 1 & 1 & 5 \\ 0 & 3 & 11 \end{array}\right] R1-\tfrac{1}{3}R2 \sim \left[\begin{array}{cc:c} 1 & 0 & 4/3 \\ 0 & 3 & 11 \end{array}\right] \tfrac{1}{3}R2 \sim \left[\begin{array}{cc:c} 1 & 0 & 4/3 \\ 0 & 1 & 11/3 \end{array}\right]$$

reduced row echelon form

- R.R.E.F. $\Rightarrow \begin{cases} c_1 = 4/3 \\ c_2 = 11/3 \end{cases} \Rightarrow$ $\boxed{\vec{x}(t) = \frac{4}{3} e^{-t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \frac{11}{3} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

Example Solve the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = -3x - 3y - z & x(0) = 5 \\ \dfrac{dy}{dt} = x + y + z & y(0) = 1 \\ \dfrac{dz}{dt} = x + 3y + z & z(0) = 5 \end{cases}$$

Solution $\Leftrightarrow$ Matrix-Vector form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}.$$

Coefficient Matrix: $A = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$.

- Eigenvalues $\det(A-\lambda I) = \det \begin{bmatrix} -3-\lambda & -3 & -1 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{bmatrix}$

Coefficient Matrix: $A = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$

- Eigenvalues

$$\det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & -3 & -1 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{bmatrix}$$ (Cofactor expansion along Column 1)

$$= (-3-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} - (1)\det\begin{bmatrix} -3 & -1 \\ 3 & 1-\lambda \end{bmatrix} + (1)\det\begin{bmatrix} -3 & -1 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (-3-\lambda)\left[(1-\lambda)^2 - 3\right] - \left[-3(1-\lambda) + 3\right] + \left[-3 + (1-\lambda)\right]$$

$$= (-3-\lambda)(\lambda^2 - 2\lambda - 2) - (3\lambda) + (-2-\lambda)$$

$$= (-3\lambda^2 + 6\lambda + 6 - \lambda^3 + 2\lambda^2 + 2\lambda) - (3\lambda) + (-2-\lambda) = \underbrace{-\lambda^3 - \lambda^2 + 4\lambda + 4}_{= f(\lambda)}.$$

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$$= (-3-\lambda)(\lambda^2-2\lambda-2) - (3\lambda) + (-2-\lambda)$$

$$= (-3\lambda^2+6\lambda+6-\lambda^3+2\lambda^2+2\lambda) - (3\lambda) + (-2-\lambda) = -\lambda^3-\lambda^2+4\lambda+4 = f(\lambda).$$

Candidates of Rational Roots: $\dfrac{(\text{an integer factor of } 4)}{(\text{an integer factor of } -1)}$,

i.e. $\pm\frac{1}{1}$, $\pm\frac{2}{1}$, $\pm\frac{4}{1}$.

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i.e. $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}$.

$f(1) = 6$, $\underbrace{f(2) = 0}_{\text{a root}}$, $f(4) = -60$, $\underbrace{f(-1) = 0}_{\text{a root}}$, $\underbrace{f(-2) = 0}_{\text{a root}}$, $f(-4) = 36$

Eigenvalues of A are: $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -2$.

$$A = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$ Eigenvalues $\lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -2$

Eigenvectors for $\lambda_1 = 2$: $(A - \lambda_1 I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \iff (A - 2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$

$$\iff \begin{bmatrix} -5 & -3 & -1 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} -5 & -3 & -1 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \underset{R1 \leftrightarrow R2}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ -5 & -3 & -1 \\ 1 & 3 & -1 \end{bmatrix} \underset{\substack{R2+5R1 \\ R3-R1}}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -8 & 4 \\ 0 & 4 & -2 \end{bmatrix} \underset{-\frac{1}{8}R2}{\sim}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 4 & -2 \end{bmatrix} \underset{\substack{R1+R2 \\ R3-4R2}}{\sim} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\iff \begin{cases} x + \frac{1}{2}z = 0 \\ y - \frac{1}{2}z = 0 \end{cases} \iff \begin{cases} x = -\frac{1}{2}z \\ y = \frac{1}{2}z \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}z \\ \frac{1}{2}z \\ z \end{bmatrix} = z\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \qquad \lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

Eigenvectors for $\lambda_2 = -1$: $(A - \lambda_2 I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \Leftrightarrow (A + I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} -2 & -3 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \overset{R1 \Leftrightarrow R2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} R2+2R1 \\ R3-R1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} R1-2R2 \\ \\ R3-R1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x - z = 0 \\ y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = -z \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \qquad \lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

Eigenvectors for $\lambda_3 = -2$: $(A - \lambda_3 I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \iff (A + 2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$

$$\iff \begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 1 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 1 \\ 1 & 3 & 3 \end{bmatrix} \begin{matrix} -R1 \\ R2+R1 \\ R3+R1 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{2}R3 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1-R3 \\ R2 \\ \Leftrightarrow R3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\iff \begin{cases} x + 3y = 0 \\ z = 0 \end{cases} \iff \begin{cases} x = -3y \\ z = 0 \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues of A are: $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -2$.

with eigenvectors $\vec{u}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

- General Solutions of $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

- Init Cond $\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \Leftrightarrow c_1\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} -1/2 & 1 & -3 \\ 1/2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$$

Row Reduce

$$\left[\begin{array}{ccc|c} -1/2 & 1 & -3 & 5 \\ 1/2 & -1 & 1 & 1 \\ 1 & 1 & 0 & 5 \end{array}\right] \begin{array}{l} -2R1 \\ 2R2 \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 6 & -10 \\ 1 & -2 & 2 & 2 \\ 1 & 1 & 0 & 5 \end{array}\right] \begin{array}{l} R2-R1 \\ R3-R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 6 & -10 \\ 0 & 0 & -4 & 12 \\ 0 & 3 & -6 & 15 \end{array}\right] \begin{array}{l} -\frac{1}{4}R2 \\ \frac{1}{3}R3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 6 & -10 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & -2 & 5 \end{array}\right] \begin{array}{l} R1+2R3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & -2 & 5 \end{array}\right] \begin{array}{l} R1-2R2 \\ \\ R3+2R2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 \end{array}\right] R2 \leftrightarrow R3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$ (reduced row echelon form)

$$\Leftrightarrow \begin{cases} c_1 = 6 \\ c_2 = -1 \\ c_3 = -3 \end{cases}$$

Answer

Solution of the Initial Value Problem

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = 6e^{2t}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 3e^{-2t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Alternative Form of the Answer

$$x(t) = -3e^{2t} - e^{-t} + 9e^{-2t}$$

$$y(t) = 3e^{2t} + e^{-t} - 3e^{-2t}$$

$$z(t) = 6e^{2t} - e^{-t}$$

Example Solve $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 3 \\ 1 & 1 & -2 \end{bmatrix}$.

Solution

- **Eigenvalues:** $\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & -3 \\ -1 & -\lambda & 3 \\ 1 & 1 & -2-\lambda \end{bmatrix}$ (cofactor expansion along Column 2)

$$= -(1)\det\begin{bmatrix} -1 & 3 \\ 1 & -2-\lambda \end{bmatrix} + (-\lambda)\det\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} - (1)\det\begin{bmatrix} 2-\lambda & -3 \\ -1 & 3 \end{bmatrix}$$

$$= -(2+\lambda-3) - \lambda(-4+\lambda^2+3) - (6-3\lambda-3) = -\lambda^3+3\lambda-2 = f(\lambda).$$

Candidates of Rational Roots:

$$\frac{(\text{an integer factor of } -2)}{(\text{an integer factor of } -1)}. \qquad \pm\frac{1}{1}, \ \pm\frac{2}{1}.$$

$$\underbrace{f(1)=0}_{\text{a root}}, \quad f(2) = -4, \quad f(-1) = -4, \quad \underbrace{f(-2)=0}_{\text{a root}}$$

$$(1) + (-2) + (\text{The 3rd root}) = \text{Trace}(A) = 2+0+(-2) = 0$$

$$\Rightarrow (\text{The 3rd root}) = 1$$

$\lambda_1 = -2$, $\quad \lambda_2 = \lambda_3 = 1$ (repeated, algebraic multiplicity 2)

$$A=\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 3 \\ 1 & 1 & -2 \end{bmatrix} \qquad \lambda_1=-2, \qquad \lambda_2=\lambda_3=1$$

- Eigenvectors for $\lambda_1=-2$: $(A+2I)\vec{x}=\vec{0} \iff \begin{bmatrix} 4 & 1 & -3 \\ -1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & 1 & -3 \\ -1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} R1 \Leftrightarrow R3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ 4 & 1 & -3 \end{bmatrix} \begin{matrix} \\ R2+R1 \\ R3-4R1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}R2 \\ \frac{1}{3}R3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R1-R2 \\ \\ R3+R2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\iff \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A=\begin{bmatrix}2&1&-3\\-1&0&3\\1&1&-2\end{bmatrix} \qquad \lambda_1=-2, \qquad \lambda_2=\lambda_3=1$$

- Eigenvectors for $\lambda_2=\lambda_3=1$: $(A-I)\vec{x}=\vec{0} \Leftrightarrow \begin{bmatrix}1&1&-3\\-1&-1&3\\1&1&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1&1&-3\\-1&-1&3\\1&1&-3\end{bmatrix}\begin{matrix}R2+R1\\R3-R1\end{matrix} \sim \begin{bmatrix}1&1&-3\\0&0&0\\0&0&0\end{bmatrix}$$

$$\Leftrightarrow x_1+x_2-3x_3=0$$

$$\Leftrightarrow x_1=-x_2+3x_3$$

$$\Leftrightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_2+3x_3\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}3\\0\\1\end{bmatrix}$$

Eigenvalues: $\lambda_1=-2$, $\lambda_2=\lambda_3=1$

Eigenvectors: $\vec{u}_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\vec{u}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\vec{u}_3=\begin{bmatrix}3\\0\\1\end{bmatrix}$

General Solutions of $\frac{d\vec{x}}{dt}=A\vec{x}$:

$$\boxed{\vec{x}(t)=c_1e^{-2t}\begin{bmatrix}1\\-1\\1\end{bmatrix}+c_2e^{t}\begin{bmatrix}-1\\1\\0\end{bmatrix}+c_3e^{t}\begin{bmatrix}3\\0\\1\end{bmatrix}}$$