n-dim. Homog. Lin. Systems with Const. Coefficients
— Repeated Eigenvalues

$\boxed{\vec{x}\,' = A\vec{x}}$

- unknown solution $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$
- A : $n \times n$ const. matrix.

How to generate solutions in the case of repeated eigenvalues?

- Suppose: $\det(A - \lambda I)$ has a root λ_0 of multiplicity k.
- Then, by a theorem in linear algebra, $(A - \lambda_0 I)^k \vec{x} = \vec{0}$ has k dim. solution space.
- Pick a basis of the sol. space for eq. $(A - \lambda_0 I)^k \vec{x} = 0$:

  $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.

- $$\begin{cases} e^{\lambda_0 t}\left\{ I + t(A-\lambda_0 I) + \frac{t^2}{2!}(A-\lambda_0 I)^2 + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda_0 I)^{k-1} \right\} \vec{u}_1, \\ \vdots \\ e^{\lambda_0 t}\left\{ I + t(A-\lambda_0 I) + \frac{t^2}{2!}(A-\lambda_0 I)^2 + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda_0 I)^{k-1} \right\} \vec{u}_k \end{cases}$$

give k linearly indep. solutions of $\vec{x}\,' = A\vec{X}$.

For example, when λ_0 is a repeated eigenvalue of multiplicity 3,

[The generalized eigenspace of λ_0]

$=$ [The solution space of $(A-\lambda_0 I)^3 \vec{x} = \vec{0}$] is 3 dimensional,

$= \mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

$\Downarrow$

3 linearly independent solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\begin{cases} e^{\lambda_0 t}\left\{ I + t(A-\lambda_0 I) + \frac{t^2}{2!}(A-\lambda_0 I)^2 \right\} \vec{u}_1, \\ e^{\lambda_0 t}\left\{ I + t(A-\lambda_0 I) + \frac{t^2}{2!}(A-\lambda_0 I)^2 \right\} \vec{u}_2, \\ e^{\lambda_0 t}\left\{ I + t(A-\lambda_0 I) + \frac{t^2}{2!}(A-\lambda_0 I)^2 \right\} \vec{u}_3. \end{cases}$$

When $\begin{cases} A \text{ is an } n \times n \text{ matrix and} \\ \lambda_0 \text{ is a repeated eigenvalue of multiplicity } n, \end{cases}$

i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ all eigenvalues are equal

by Cayley-Hamilton Theorem (a theorem in linear algebra),

we always have $(A - \lambda_0 I)^n = 0$, and hence

[The generalized eigenspace of λ_0]

$= [\text{The Solution Space of } (A - \lambda_0 I)^n \vec{x} = \vec{0}]$ is n-dimensional

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots\cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Span}\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$$

$$\Downarrow$$

$$\begin{cases} e^{\lambda_0 t}\left\{ I + t(A - \lambda_0 I) + \dfrac{t^2}{2!}(A - \lambda_0 I)^2 + \cdots + \dfrac{t^{n-1}}{(n-1)!}(A - \lambda_0 I)^{n-1} \right\} \vec{e}_1, \\ \vdots \\ e^{\lambda_0 t}\left\{ I + t(A - \lambda_0 I) + \dfrac{t^2}{2!}(A - \lambda_0 I)^2 + \cdots + \dfrac{t^{n-1}}{(n-1)!}(A - \lambda_0 I)^{n-1} \right\} \vec{e}_n \end{cases}$$

give n linearly indep. solutions of $\dfrac{d\vec{x}}{dt} = A\vec{x}$.

In other words, the columns of $e^{\lambda_0 t}\left\{ I + t(A - \lambda_0 I) + \dfrac{t^2}{2!}(A - \lambda_0 I)^2 + \cdots + \dfrac{t^{n-1}}{(n-1)!}(A - \lambda_0 I)^{n-1} \right\}$ give n lin. indep. sols

Derivation of the Formula When λ_0 has multiplicity 3

- Look for sol's of the form $\vec{x}(t) = e^{\lambda_0 t}\{\vec{u} + t\vec{v} + t^2\vec{w}\}$.
- Substitue in the diff eq: $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\lambda_0 e^{\lambda_0 t}\{\vec{u} + t\vec{v} + t^2\vec{w}\} + e^{\lambda_0 t}\{\vec{v} + 2t\vec{w}\} = e^{\lambda_0 t}\{A\vec{u} + tA\vec{v} + t^2 A\vec{w}\}$$

- Compare the Coefficients on two sides :

$$\left.\begin{array}{lll} \text{Coeff. of } e^{\lambda_0 t} & \Rightarrow & \lambda_0\vec{u} + \vec{v} = A\vec{u} \\ \text{Coeff. of } te^{\lambda_0 t} & \Rightarrow & \lambda_0\vec{v} + 2\vec{w} = A\vec{v} \\ \text{Coeff. of } t^2 e^{\lambda_0 t} & \Rightarrow & \lambda_0\vec{w} = A\vec{w} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \vec{v} = (A-\lambda_0 I)\vec{u} \\ \vec{w} = \frac{1}{2}(A-\lambda_0 I)\vec{v} \\ (A-\lambda_0 I)\vec{w} = \vec{0} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \vec{v} = (A-\lambda_0 I)\vec{u} \\ \vec{w} = \frac{1}{2}(A-\lambda_0 I)^2\vec{u} \\ (A-\lambda_0 I)\vec{w} = \vec{0} \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \vec{v} = (A-\lambda_0 I)\vec{u} \\ \vec{w} = \frac{1}{2}(A-\lambda_0 I)^2\vec{u} \\ \frac{1}{2}(A-\lambda_0 I)^3\vec{u} = \vec{0} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (A-\lambda_0 I)^3\vec{u} = \vec{0} \\ \vec{v} = (A-\lambda_0 I)\vec{u} \\ \vec{w} = \frac{1}{2}(A-\lambda_0 I)^2\vec{u} \end{array}\right.$$

That is: $\vec{x}(t) = e^{\lambda_0 t}\{\vec{u} + t\vec{v} + t^2\vec{w}\}$ Satisfies $\frac{d\vec{x}}{dt} = A\vec{x}$

if and only if $(A-\lambda_0 I)^3\vec{u} = \vec{0}$, $\vec{v} = (A-\lambda_0 I)\vec{u}$, $\vec{w} = \frac{1}{2}(A-\lambda_0 I)^2\vec{u}$

Equivalently,

$$\vec{x}(t) = e^{\lambda_0 t}\{\vec{u} + t(A-\lambda_0 I)\vec{u} + \frac{t^2}{2}(A-\lambda_0 I)^2\vec{u}\} = e^{\lambda_0 t}\{I + t(A-\lambda_0 I) + \frac{t^2}{2}(A-\lambda_0 I)^2\}\vec{u},$$

with $(A-\lambda_0 I)^3\vec{u} = \vec{0}$

Example 1 Solve $\vec{x}\,' = A\vec{x}$, where $A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix}$.

$\boxed{\text{Solution}}$

- **Eigenvalues:** $\det(A - \lambda I) = \cdots \text{ expand } \cdots = -\lambda^3 - 7\lambda^2 - 15\lambda - 9$

$$= -(\lambda+1)(\lambda+3)^2 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = \lambda_3 = -3.$$

- **Eigenvectors for $\lambda_1 = -1$**

$$(A+I)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -4 & -8 & 4 \\ 2 & 4 & -2 \\ 6 & 14 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

- **Generalized Eigenvectors for $\lambda_2 = \lambda_3 = -3$**

$$(A+3I)^2\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 12 & 24 & 0 \\ -4 & -8 & 0 \\ 4 & 8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 + 2x_2 = 0$$

$$\Leftrightarrow x_1 = -2x_2. \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $e^{-3t}\left\{I + t(A+3I)\right\}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = e^{-3t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -2 & -8 & 4 \\ 2 & 6 & -2 \\ 6 & 14 & -2 \end{bmatrix}\right\}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = e^{-3t}\begin{bmatrix} -2-4t \\ 1+2t \\ 2t \end{bmatrix}$

- $e^{-3t}\left\{I + t(A+3I)\right\}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{-3t}\left\{ \text{------} \right\}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{-3t}\begin{bmatrix} 4t \\ -2t \\ 1-2t \end{bmatrix}$

- **Gen. Sol's of $\vec{x}\,' = A\vec{x}$:**

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} -2-4t \\ 1+2t \\ 2t \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 4t \\ -2t \\ 1-2t \end{bmatrix}$$

Example 2 Solve $\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix}$

Solution

- Eigenvalues: $\det(A - \lambda I) = \det\begin{bmatrix} -7-\lambda & -9 & 9 \\ 3 & 5-\lambda & -3 \\ -3 & -3 & 5-\lambda \end{bmatrix} = \cdots = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda-2)^2$

$\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 2$ (repeated, multiplicity 2)

- Eigenvectors for $\lambda_1 = -1$

$$(A+I)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -6 & -9 & 9 \\ 3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

- Generalized Eigenvectors for $\lambda_2 = \lambda_3 = 2$

$$(A-2I)^2\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 27 & 27 & -27 \\ -9 & -9 & 9 \\ 9 & 9 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 + x_2 - x_3 = 0$$

$$\Leftrightarrow x_1 = -x_2 + x_3 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- $$e^{2t}\{I + t(A-2I)\}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -9 & -9 & 9 \\ 3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix}\right\}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

- $$e^{2t}\{I + t(A-2I)\}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = e^{2t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -9 & -9 & 9 \\ 3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

- Gen. Sol's of $\vec{x}' = A\vec{x}$:

$$\vec{x}(t) = C_1 e^{-t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Example 3 Solve $\vec{x}\,' = A\vec{x}$, where $A = \begin{bmatrix} 11/2 & 1 & -1/2 \\ -1 & 7 & 1 \\ 1/2 & -1 & 17/2 \end{bmatrix}$.

$\boxed{\text{Solution}}$

- Eigenvalues: $\det(A-\lambda I) = \det\begin{bmatrix} \frac{11}{2}-\lambda & 1 & -\frac{1}{2} \\ -1 & 7-\lambda & 1 \\ \frac{1}{2} & -1 & \frac{17}{2}-\lambda \end{bmatrix} = \dots = -\lambda^3 + 21\lambda^2 - 147\lambda + 343$

$= -(\lambda-7)^3$.

$\lambda_1 = \lambda_2 = \lambda_3 = 7$ (repeated, multiplicity 3).

- The columns of $e^{7t}\left\{ I + t(A-7I) + \frac{t^2}{2!}(A-7I)^2 \right\}$ give 3 lin. indep. sols

$$= e^{7t}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -3/2 & 1 & -1/2 \\ -1 & 0 & 1 \\ 1/2 & -1 & 3/2 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \right\}$$

$$= e^{7t}\begin{bmatrix} 1-\frac{3}{2}t+\frac{1}{2}t^2 & t-\frac{1}{2}t^2 & -\frac{1}{2}t+\frac{1}{2}t^2 \\ -t+t^2 & 1-t^2 & t+t^2 \\ \frac{1}{2}t+\frac{1}{2}t^2 & -t-\frac{1}{2}t^2 & 1+\frac{3}{2}t+\frac{1}{2}t^2 \end{bmatrix}$$

- Gen. Solutions of $\vec{x}\,' = A\vec{x}$

$$\vec{x}(t) = C_1 e^{7t}\begin{bmatrix} 1-\frac{3}{2}t+\frac{1}{2}t^2 \\ -t+t^2 \\ \frac{1}{2}t+\frac{1}{2}t^2 \end{bmatrix} + C_2 e^{7t}\begin{bmatrix} t-\frac{1}{2}t^2 \\ 1-t^2 \\ -t-\frac{1}{2}t^2 \end{bmatrix} + C_3 e^{7t}\begin{bmatrix} -\frac{1}{2}t+\frac{1}{2}t^2 \\ t+t^2 \\ 1+\frac{3}{2}t+\frac{1}{2}t^2 \end{bmatrix}$$

Example 4 Solve $\vec{x}\,' = A\vec{x}$, where $A = \begin{bmatrix} 4 & -4 & -5 & 4 & -3 \\ 4 & 0 & -4 & 2 & 1 \\ 3 & -5 & -4 & 5 & -4 \\ 0 & -1 & 0 & 3 & -1 \\ -5 & 2 & 5 & -2 & 1 \end{bmatrix}$.

Solution

Eigenvalues $\det(A - \lambda I)$

$$= \det\begin{bmatrix} 4-\lambda & -4 & -5 & 4 & -3 \\ 4 & -\lambda & -4 & 2 & 1 \\ 3 & -5 & -4-\lambda & 5 & -4 \\ 0 & -1 & 0 & 3-\lambda & -1 \\ -5 & 2 & 5 & -2 & 1-\lambda \end{bmatrix} = \det\begin{bmatrix} 4-\lambda & -4 & -5 & -8+4\lambda & 1 \\ 4 & -\lambda & -4 & 2-3\lambda+\lambda^2 & 1+\lambda \\ 3 & -5 & -4-\lambda & -10+5\lambda & 1 \\ 0 & -1 & 0 & 0 & 0 \\ -5 & 2 & 5 & 4-2\lambda & -1-\lambda \end{bmatrix} = -\det\begin{bmatrix} 4-\lambda & -5 & 4(\lambda-2) & 1 \\ 4 & -4 & (\lambda-2)(\lambda-1) & 1+\lambda \\ 3 & -4-\lambda & 5(\lambda-2) & 1 \\ -5 & 5 & -2(\lambda-2) & -1-\lambda \end{bmatrix}$$

$$= -(\lambda-2)\det\begin{bmatrix} 4-\lambda & -5 & 4 & 1 \\ 4 & -4 & \lambda-1 & 1+\lambda \\ 3 & -4-\lambda & 5 & 1 \\ -5 & 5 & -2 & -1-\lambda \end{bmatrix} = -(\lambda-2)\det\begin{bmatrix} -1-\lambda & -5 & 4 & 1 \\ 0 & -4 & \lambda-1 & 1+\lambda \\ -1-\lambda & -4-\lambda & 5 & 1 \\ 0 & 5 & -2 & -1-\lambda \end{bmatrix} = -(\lambda-2)(-1-\lambda)\det\begin{bmatrix} 1 & -5 & 4 & 1 \\ 0 & -4 & \lambda-1 & 1+\lambda \\ 1 & -4-\lambda & 5 & 1 \\ 0 & 5 & -2 & -1-\lambda \end{bmatrix}$$

$$= (\lambda-2)(\lambda+1)\det\begin{bmatrix} 1 & -5 & 4 & 1 \\ 0 & -4 & \lambda-1 & 1+\lambda \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 5 & -2 & -1-\lambda \end{bmatrix} = (\lambda-2)(\lambda+1)\det\begin{bmatrix} -4 & \lambda-1 & 1+\lambda \\ 1-\lambda & 1 & 0 \\ 5 & -2 & -1-\lambda \end{bmatrix} = (\lambda-2)(\lambda+1)^2\det\begin{bmatrix} -4 & \lambda-1 & 1 \\ 1-\lambda & 1 & 0 \\ 5 & -2 & -1 \end{bmatrix}$$

$$= (\lambda-2)(\lambda+1)^2\det\begin{bmatrix} -4 & \lambda-1 & 1 \\ 1-\lambda & 1 & 0 \\ 1 & \lambda-3 & 0 \end{bmatrix} = (\lambda-2)(\lambda+1)^2\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & \lambda-3 \end{bmatrix} = (\lambda-2)(\lambda+1)^2(-\lambda^2+4\lambda-4)$$

$$= (\lambda-2)(\lambda+1)^2\left[-(\lambda-2)^2\right] = -(\lambda-2)^3(\lambda+1)^2.$$

$\lambda_1 = \lambda_2 = -1$ (multiplicity 2), $\lambda_3 = \lambda_4 = \lambda_5 = 2$ (multiplicity 3).

· Generalized Eigenspace for $\lambda_1 = \lambda_2 = -1$:

$$(A+I)^2 \vec{x} = \vec{0} \iff \begin{bmatrix} 9 & -9 & -9 & 9 & -9 \\ 7 & 5 & -7 & 4 & 5 \\ 6 & -15 & -6 & 15 & -15 \\ 1 & -7 & -1 & 16 & -7 \\ -12 & 3 & 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_5 = 0 \\ x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 \\ x_2 = -x_5 \\ x_4 = 0 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

· Two Linearly Indep. Sol's of $\vec{x}' = A\vec{x}$

- $$e^{-t}\{I + t(A+I)\} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = e^{-t}\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 5 & -4 & -5 & 4 & -3 \\ 4 & 1 & -4 & 2 & 1 \\ 3 & -5 & -3 & 5 & -4 \\ 0 & -1 & 0 & 4 & -1 \\ -5 & 2 & 5 & -2 & 2 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

- $$e^{-t}\{I + t(A+I)\} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = e^{-t}\{ \cdots\cdots \} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} t \\ -1 \\ t \\ 0 \\ 1 \end{bmatrix}$$

· Generalized Eigenspace for $\lambda_3 = \lambda_4 = \lambda_5 = 2$:

$$(A-2I)^3 \vec{x} = \vec{0} \iff \begin{bmatrix} 54 & -54 & -81 & 54 & -27 \\ 54 & -27 & -54 & 27 & 0 \\ 54 & -54 & -81 & 54 & -27 \\ 0 & 0 & 0 & 0 & 0 \\ -54 & 27 & 54 & -27 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_5 = 0 \\ x_2 + x_3 - x_4 + x_5 = 0 \end{cases} \iff \begin{cases} x_1 = \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ x_2 = -x_3 + x_4 - x_5 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ -x_3 + x_4 - x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

· Three Linearly Indep. Sols of $\vec{x}' = A\vec{x}$

- $$e^{2t}\left\{ I + t(A-2I) + \frac{t^2}{2!}(A-2I)^2 \right\} \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= e^{2t}\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 2 & -4 & -5 & 4 & -3 \\ 4 & -2 & -4 & 2 & 1 \\ 3 & -5 & -6 & 5 & -4 \\ 0 & -1 & 0 & 1 & -1 \\ -5 & 2 & 5 & -2 & -1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -12 & 15 & 21 & -15 & 9 \\ -17 & 8 & 17 & -8 & -1 \\ -12 & 15 & 21 & -15 & 9 \\ 1 & -1 & -1 & 1 & -1 \\ 18 & -9 & -18 & 9 & 0 \end{bmatrix} \right\} \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} \frac{1}{2} \\ -1 + \frac{1}{4}t^2 \\ 1 + \frac{1}{2}t \\ t + \frac{1}{4}t^2 \\ \frac{1}{2}t \end{bmatrix}$$

- $$e^{2t}\left\{ \cdots\cdots \right\} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

- $$e^{2t}\left\{ \cdots\cdots \right\} \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} -\frac{1}{2} \\ -1 + t - \frac{1}{4}t^2 \\ -\frac{1}{2}t \\ -\frac{1}{4}t^2 \\ 1 - \frac{1}{2}t \end{bmatrix}$$

• General Solutions of $\vec{x}' = A\vec{x}$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$$

$$= c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} t \\ -1 \\ t \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} \frac{1}{2} \\ -1+\frac{1}{4}t^2 \\ 1+\frac{1}{2}t \\ t+\frac{1}{4}t^2 \\ \frac{1}{2}t \end{bmatrix} + c_4 e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_5 e^{2t}\begin{bmatrix} -\frac{1}{2} \\ -1+t-\frac{1}{4}t^2 \\ -\frac{1}{2}t \\ -\frac{1}{4}t^2 \\ 1-\frac{1}{2}t \end{bmatrix}$$