Complex Eigenvalues in n-D Homogeneous Linear Systems with Constant Coefficients

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where } \vec{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A: \text{ } n\times n \text{ constant real matix}$$

- Suppose we have

complex eigenvalues $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$

eigenvector for $\lambda = \alpha + \beta i$

$$\vec{u} = \vec{v} + i\vec{w} \quad \text{where } \vec{v} = \mathrm{Re}\,\vec{u},\ \vec{w} = \mathrm{Im}\,\vec{u}.$$

$\Downarrow$

- Solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ in the Complex Form

$$e^{(\alpha+\beta i)t}(\vec{v} + i\vec{w}), \quad e^{(\alpha-\beta i)t}(\vec{v} - i\vec{w})$$

- Solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ in the Real Form

$$\mathrm{Re}\left\{e^{(\alpha+\beta i)t}(\vec{v} + i\vec{w})\right\}, \quad \mathrm{Im}\left\{e^{(\alpha+\beta i)t}(\vec{v} + i\vec{w})\right\}$$

or, equivalently,

$$e^{\alpha t}\left\{\cos(\beta t)\vec{v} - \sin(\beta t)\vec{w}\right\},$$

$$e^{\alpha t}\left\{\sin(\beta t)\vec{v} + \cos(\beta t)\vec{w}\right\}.$$

Example Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -26 \\ 4 & 5 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution

• Eigenvalues: $\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & -26 \\ 4 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) - (-26)(4)$

$= \lambda^2 - 6\lambda + 109 = 0$

$\Leftrightarrow (\lambda-3)^2 + 100 = 0 \Leftrightarrow (\lambda-3)^2 = -100 \Leftrightarrow \lambda - 3 = \pm 10i$

$\Leftrightarrow \lambda_{1,2} = 3 \pm 10i$

• Eigenvectors for $\lambda_1 = 3+10i$:

$(A-\lambda_1 I)\vec{x} = \vec{0} \Leftrightarrow (A-(3+10i)I)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -2-10i & -26 \\ 4 & 2-10i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow 4x_1 + (2-10i)x_2 = 0$

$\Leftrightarrow x_1 = \frac{-2+10i}{4} x_2 = \left(-\frac{1}{2} + \frac{5}{2}i\right) x_2$

$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{5}{2}i\right)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2}+\frac{5}{2}i \\ 1 \end{bmatrix}$

An eigenvector $\vec{u}_1 = \begin{bmatrix} -\frac{1}{2}+\frac{5}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + i \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}$

• Gen. Solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\vec{x}(t) = c_1 e^{3t}\left\{\cos(10t)\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix} - \sin(10t)\begin{bmatrix}\frac{5}{2}\\0\end{bmatrix}\right\}$$

$$+ c_2 e^{3t}\left\{\sin(10t)\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix} + \cos(10t)\begin{bmatrix}\frac{5}{2}\\0\end{bmatrix}\right\}$$

• Init. Cond. $\vec{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}$

$$\Leftrightarrow \quad c_1\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix} + c_2\begin{bmatrix}\frac{5}{2}\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \Leftrightarrow \begin{cases}-\frac{1}{2}c_1 + \frac{5}{2}c_2 = 1\\ c_1 = 2\end{cases} \Leftrightarrow \begin{cases}c_1 = 2\\ c_2 = \frac{4}{5}\end{cases}$$

• Solution of the initial value problem :

$$\boxed{\begin{aligned}\vec{x}(t) &= 2e^{3t}\left\{\cos(10t)\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix} - \sin(10t)\begin{bmatrix}\frac{5}{2}\\0\end{bmatrix}\right\}\\ &+ \frac{4}{5}e^{3t}\left\{\sin(10t)\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix} + \cos(10t)\begin{bmatrix}\frac{5}{2}\\0\end{bmatrix}\right\}\end{aligned}}$$

Example Solve $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -5 & -6 & 12 \\ 3 & 7 & -12 \\ 0 & 3 & -5 \end{bmatrix} \vec{x}$, $\quad \vec{x}(0) = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$.

Solution

· Eigenvalues: $\det(A-\lambda I) = \det \begin{bmatrix} -5-\lambda & -6 & 12 \\ 3 & 7-\lambda & -12 \\ 0 & 3 & -5-\lambda \end{bmatrix}$ (Cofactor Expansion along Column 1)

$$= (-5-\lambda)\det\begin{bmatrix} 7-\lambda & -12 \\ 3 & -5-\lambda \end{bmatrix} - (3)\det\begin{bmatrix} -6 & 12 \\ 3 & -5-\lambda \end{bmatrix} + 0$$

$$= (-5-\lambda)(\lambda^2-2\lambda+1) - 3(6\lambda-6) = (-5\lambda^2+10\lambda-5-\lambda^3+2\lambda^2-\lambda) - 18\lambda+18$$

$$= -\lambda^3 - 3\lambda^2 - 9\lambda + 13 = f(\lambda)$$

Candidates of Rational Roots: $\dfrac{(\text{an integer factor of } 13)}{(\text{an integer factor of } 1)}$

$$\pm\frac{1}{1}, \pm\frac{13}{1}.$$

$f(1) = 0$, $\quad f(-1) = 20$, $\quad f(13) = -13^3 - 3(13^2) - 9(13) + 13 < 0$

a root

$f(-13) = 13^3 - 3(13^2) + 9(13) + 13 > 0$

• $\lambda_1 = 1$ is a root of $f(\lambda) \Leftrightarrow \lambda - 1$ divides $f(\lambda) = -\lambda^3 - 3\lambda^2 - 9\lambda + 13$

Long Division

$$
\begin{array}{r}
-\lambda^2 - 4\lambda - 13 \\
\lambda - 1 \;\big)\; \overline{-\lambda^3 - 3\lambda^2 - 9\lambda + 13} \\
\underline{-\lambda^3 + \lambda^2 \qquad\qquad\qquad} \\
-4\lambda^2 - 9\lambda + 13 \\
\underline{-4\lambda^2 + 4\lambda \qquad} \\
-13\lambda + 13 \\
\underline{-13\lambda + 13} \\
0
\end{array}
$$

$$-\lambda^3 - 3\lambda^2 - 9\lambda + 13$$

$$= \underbrace{(\lambda - 1)}_{\Downarrow}\,\underbrace{(-\lambda^2 - 4\lambda - 13)} = 0$$

$$\lambda_1 = 1 \qquad \lambda_{2,3} = \frac{-4 \pm \sqrt{4^2 - 4(-1)(-13)}}{2} = -2 \pm \sqrt{4 - 13} = -2 \pm \sqrt{-9} = -2 \pm 3i$$

$A = \begin{bmatrix} -5 & -6 & 12 \\ 3 & 7 & -12 \\ 0 & 3 & -5 \end{bmatrix}$ Eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2+3i$, $\lambda_3 = -2-3i$

• Eigenvectors for $\lambda_1 = 1$: $(A - \lambda_1 I)\vec{x} = \vec{0} \Leftrightarrow (A - I)\vec{x} = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} -6 & -6 & 12 \\ 3 & 6 & -12 \\ 0 & 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -6 & 12 \\ 3 & 6 & -12 \\ 0 & 3 & -6 \end{bmatrix} \begin{matrix} -\frac{1}{6}R1 \\ \frac{1}{3}R2 \\ \frac{1}{3}R3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} R2-R1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} R1-R2 \\ \\ R3-R2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 2x_3 \end{cases} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

An Eigenvector for $\lambda_1 = 1$: $\vec{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$$A=\begin{bmatrix}-5 & -6 & 12\\ 3 & 7 & -12\\ 0 & 3 & -5\end{bmatrix} \quad \text{Eigenvalues} \quad \lambda_1=1,\ \lambda_2=-2+3i,\ \lambda_3=-2-3i$$

• Eigenvectors for $\lambda_2=-2+3i$: $(A-\lambda_2 I)\vec{x}=\vec{0} \Leftrightarrow (A-(-2+3i)I)\vec{x}=\vec{0}$

$$\Leftrightarrow \begin{bmatrix}-3-3i & -6 & 12\\ 3 & 9-3i & -12\\ 0 & 3 & -3-3i\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-3-3i & -6 & 12\\ 3 & 9-3i & -12\\ 0 & 3 & -3-3i\end{bmatrix} \text{reorder rows} \sim \begin{bmatrix}3 & 9-3i & -12\\ 0 & 3 & -3-3i\\ -3-3i & -6 & 12\end{bmatrix}\begin{array}{l}\frac{1}{3}R1\\ \frac{1}{3}R2\\ \frac{1}{3}R3\end{array}$$

$$\sim \begin{bmatrix}1 & 3-i & -4\\ 0 & 1 & -1-i\\ -1-i & -2 & 4\end{bmatrix} R3+(1+i)R1 \sim \begin{bmatrix}1 & 3-i & -4\\ 0 & 1 & -1-i\\ 0 & 2+2i & -4i\end{bmatrix}\begin{array}{l}R1-(3-i)R2\\ \\ R3-(2+2i)R2\end{array} \sim \begin{bmatrix}1 & 0 & 2i\\ 0 & 1 & -1-i\\ 0 & 0 & 0\end{bmatrix}$$

$$\Leftrightarrow \begin{cases}x_1+2ix_3=0\\ x_2+(-1-i)x_3=0\end{cases} \Leftrightarrow \begin{cases}x_1=-2ix_3\\ x_2=(1+i)x_3\end{cases} \Leftrightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2ix_3\\ (1+i)x_3\\ x_3\end{bmatrix}=x_3\begin{bmatrix}-2i\\ 1+i\\ 1\end{bmatrix}$$

Eigenvalues : $\lambda_1 = 1$, $\lambda_2 = -2+3i$, $\lambda_3 = -2-3i$

Eigenvectors : $\vec{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -2i \\ 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

· Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \left\{ \cos(3t) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \sin(3t) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$
$$+ c_3 e^{-2t} \left\{ \sin(3t) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \cos(3t) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

· Init. Cond.

$\vec{x}(0) = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$ $\Leftrightarrow$ $c_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} -2c_3 = -6 \\ 2c_1 + c_2 + c_3 = 3 \\ c_1 + c_2 = 2 \end{cases} \Leftrightarrow \begin{cases} c_3 = 3 \\ 2c_1 + c_2 = 0 \\ c_1 + c_2 = 2 \end{cases} \Leftrightarrow \begin{cases} c_3 = 3 \\ c_1 = -2 \\ c_2 = 4 \end{cases}$$

- Solution of the Initial Value Problem

$$\boxed{\begin{aligned}\vec{x}(t) = -2e^{t}\begin{bmatrix}0\\2\\1\end{bmatrix} &+ 4e^{-2t}\left\{\cos(3t)\begin{bmatrix}0\\1\\1\end{bmatrix} - \sin(3t)\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\} \\ &+ 3e^{-2t}\left\{\sin(3t)\begin{bmatrix}0\\1\\1\end{bmatrix} + \cos(3t)\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}\end{aligned}}$$