Predator Prey Systems

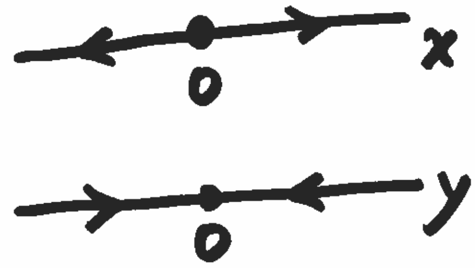
by Xu-Yan Chen

Assume No interaction

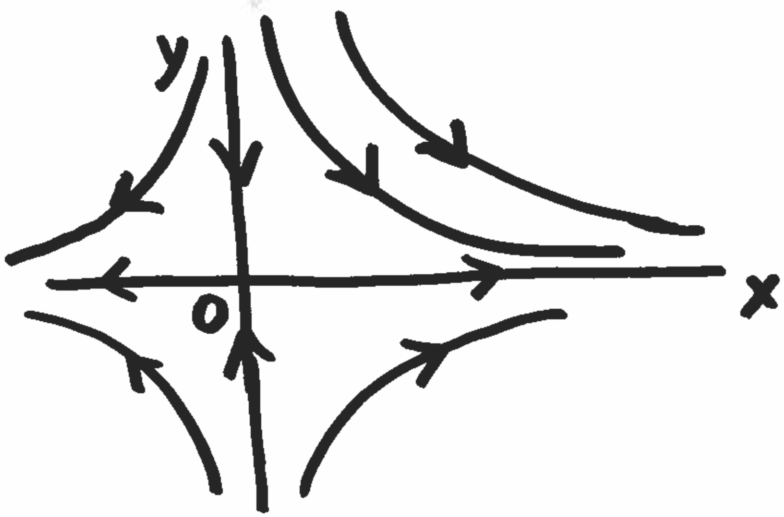
$$\begin{cases} x(t): \text{the prey population} \\ y(t): \text{the predator population} \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = -cy \end{cases}$$

Solutions: $\begin{cases} x(t) = C_1 e^{at} \\ y(t) = C_2 e^{-ct} \end{cases}$



Phase Portrait on the (x, y)-plane



For positive solutions:

$$\begin{cases} \lim\limits_{t\to\infty} x(t) = \infty \\ \lim\limits_{t\to\infty} y(t) = 0 \end{cases}$$

Predator-Prey Model

$\begin{cases} x(t): \text{the prey Population} \\ y(t): \text{the predator population} \end{cases}$

$$\left\{ \begin{aligned} \frac{dx}{dt} &= (a - \alpha y)x \\ \frac{dy}{dt} &= (-c + \gamma x)y \end{aligned} \right\} \text{Lotka-Volterra Equations}$$

[The Growth Rate of Prey] $= a - \alpha y$.

The Meaning of this Term:

The predator reduces the growth rate of the prey

[The Growth Rate of Predator] $= -c + \gamma x$.

The meaning of this term:

The prey helps the growth of the predator population

$\boxed{\text{Goal: Find the solution behavior of positive solutions.}}$

Example $\begin{cases} \frac{dx}{dt} = (1-0.5y)x \\ \frac{dy}{dt} = (-0.75+0.25x)y \end{cases}$

Find Equilibria :

$$\begin{cases} (1-0.5y)x = 0 & ① \\ (-0.75+0.25x)y = 0 & ② \end{cases}$$

① $\Rightarrow$ Either Case 1 : $x=0$

or Case 2 : $1-0.5y=0$

Case 1 : $x=0$ $\Rightarrow$ Plug in ② : $-0.75y=0$ $\Rightarrow$ $y=0$.

Case 2 : $1-0.5y=0$, $y=2$ $\Rightarrow$ Plug in ② : $-0.75+0.25x=0$, $x=3$.

Two Equilibria : $(0,0)$, $(3,2)$.

$$\begin{cases} \dfrac{dx}{dt} = (1-0.5y)x & = f(x,y) \\ \dfrac{dy}{dt} = (-0.75+0.25x)y & = g(x,y) \end{cases}$$

Equilibria:
$(0,0)$, $(3,2)$.

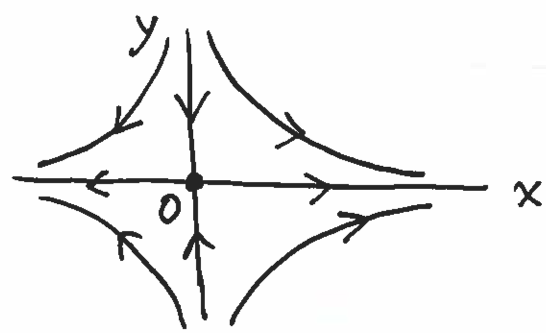
- Jacobian Matrix $\Rightarrow$ Linear Approximating System Near an Equilibrium
- $J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}$
- $J\big|_{(x,y)=(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix}$
- Lin. Approx. System Near $(x,y) \approx (0,0)$:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda_1 = 1$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = -0.75$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ a saddle, unstable.

- Lin. Approx. Dynamics
$\Leftrightarrow$ Nonlin. Dynamics Near $(x,y) \approx (0,0)$.



$$\begin{cases} \dfrac{dx}{dt} = (1-0.5y)x \\ \dfrac{dy}{dt} = (-0.75+0.25x)y \end{cases}$$

Equilibria : $(0,0)$, $(3,2)$.

- $J\big|_{(x,y)=(3,2)} = \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}\Bigg|_{(x,y)=(3,2)} = \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$

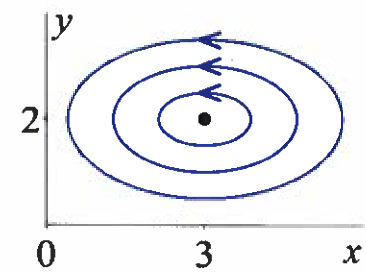
- Lin. Approx. System Near Equilibrium $(x,y) \approx (3,2)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$$

$\lambda_1 = 0.5\sqrt{3}\,i$, $\vec{u}_1 = \begin{bmatrix} \sqrt{3}i \\ 1 \end{bmatrix}$

$\lambda_2 = -0.5\sqrt{3}\,i$, $\vec{u}_2 = \begin{bmatrix} -\sqrt{3}i \\ 1 \end{bmatrix}$.

For the Lin. Approx. System.
(3,2) is a center.
is stable, but not asymp. stable.
is surrounded by periodic curves.



Phase Portrait of the Lin. Approx. System.

- $\mathrm{Re}[\lambda_{1,2}]=0$: neutral eigenvalues.

- Lin. Approx. Dynamics and Nonlinear Dynamics Near (3,2) may be the same & may be different.

Need additional analysis.

$$\begin{cases} \dfrac{dx}{dt} = (1-0.5y)x \\ \dfrac{dy}{dt} = (-0.75+0.25x)y \end{cases}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{(-0.75+0.25x)y}{(1-0.5y)x} \quad \longleftarrow \text{A separable Equation!}$$

$$\frac{1-0.5y}{y}\,dy = \frac{-0.75+0.25x}{x}\,dx$$

$$\left(\frac{1}{y}-0.5\right)dy = \left(-\frac{0.75}{x}+0.25\right)dx$$

$\Downarrow$ integrate

$$\ln y - 0.5y = -0.75\ln x + 0.25x + C$$

$$\Downarrow$$

$$\boxed{0.75\ln x - 0.25x + \ln y - 0.5y = C}$$

(an implicit solution formula)

- Eq. of Solution Curves on the (x, y) plane:

$$0.75 \ln x - 0.25x + \ln y - 0.5y = C.$$

$\Phi(x,y) =$

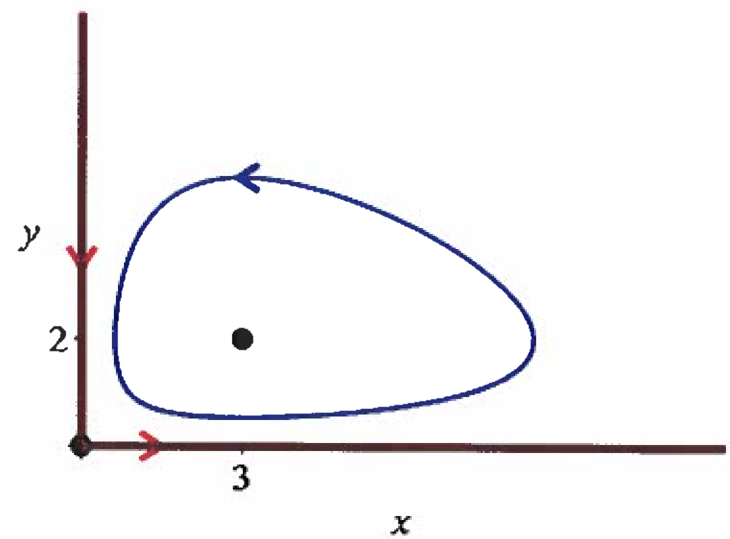
- Different values of C $\Rightarrow$ Different Solution Curves

- Example: The Solution Curve passing through $(x, y) = (3, 5)$:

$\Phi(x,y) = \Phi(3,5)$, i.e.

$$0.75 \ln x - 0.25x + \ln y - 0.5y = 0.75 \ln 3 + \ln 5 - 3.25 \approx -0.8166\cdots$$

This is a closed curve surrounding equilibrium $(3, 2)$.



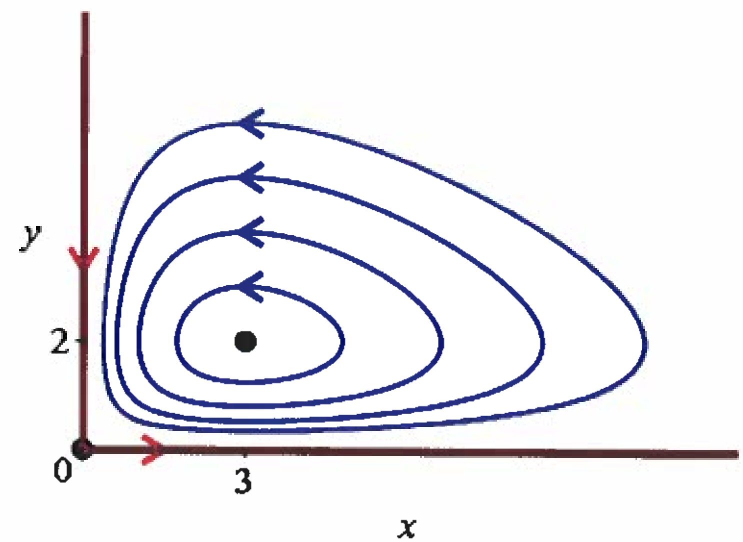
• Eq of Solution Curves on the (x,y) plane:

$$\Phi(x,y) = 0.75 \ln x - 0.25x + \ln y - 0.5y = C$$

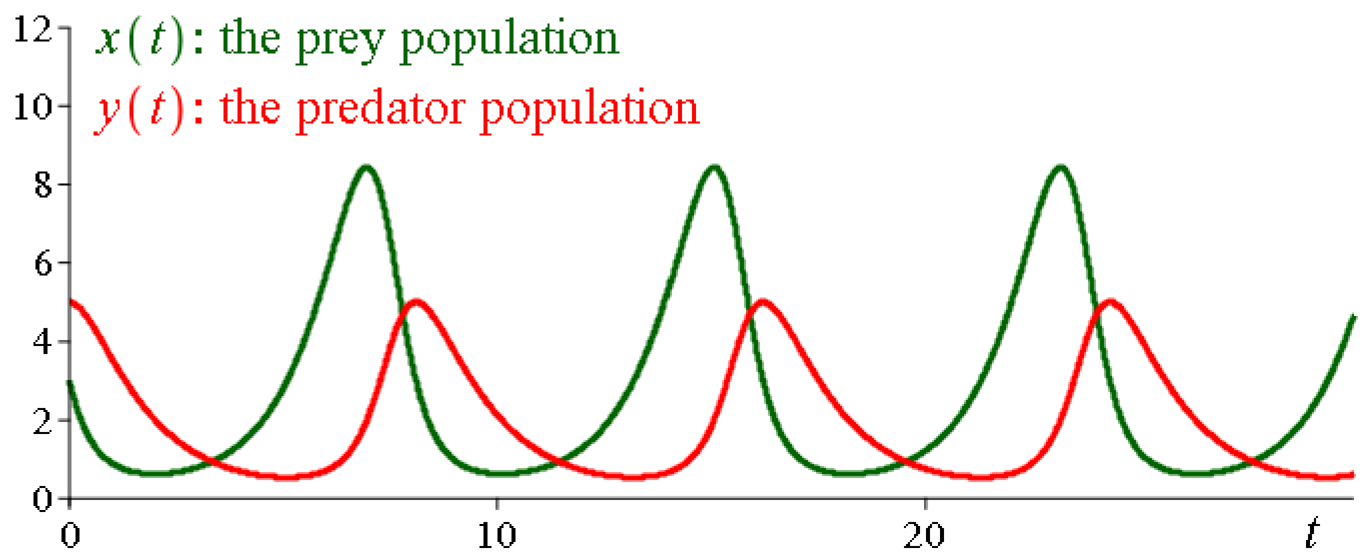
• Let $C = -0.33,\ -0.54,\ -0.82,\ -1.13$.
we get four closed curves surrounding equilibrium $(3,2)$:

- $0.75 \ln x - 0.25x + \ln y - 0.5y = -0.33$
- $0.75 \ln x - 0.25x + \ln y - 0.5y = -0.54$
- $0.75 \ln x - 0.25x + \ln y - 0.5y = -0.82$
- $0.75 \ln x - 0.25x + \ln y - 0.5y = -1.13$

- Equilibrium $(3,2)$ is a "center" with respect to the Nonlinear System,
- is stable but not asymp. stable with respect to the Nonlin. System.



Phase Portrait of the Nonlinear System



Solution of the Nonlinear system: ***x(t)*** vs ***t*** and ***y(t)*** vs ***t***