A Catalogue of Phase Portraits of 2-D Linear Flows

Consider all (real variable) solutions of 2-D system

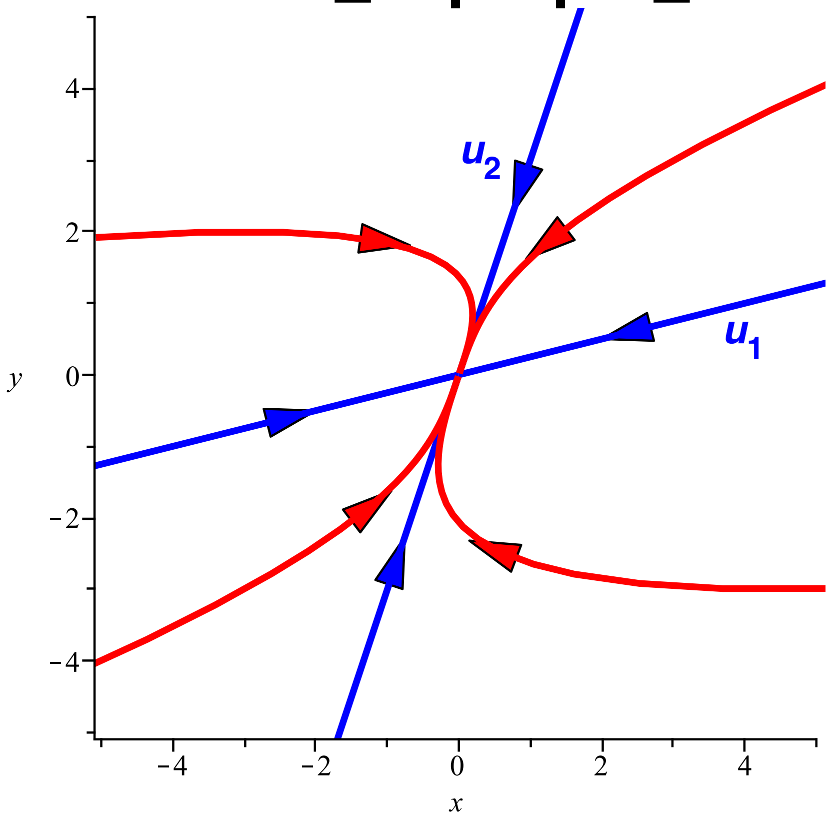
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

where A is a constant 2×2 real matrix.

By sketching the phase portrait of this system, we mean to draw several representative solution curves on the x-y plane to display typical asymptotic behavior, especially the solution behavior as $t \to \infty$ and $t \to -\infty$. Equilibria and periodic solutions should be displayed. The stability of equilibria and periodic solutions should be easily read off from the picture.

The structure of eigenvalues and (generalized) eigenvectors of matrix A gives the solution formula, completely determines dynamic behavior of the system, and therefore will also guide our classification of phase portraits.

Attractive_improper_node



The eigenvalues of A are: $\lambda_1 < \lambda_2 < 0$.

Solution Method:

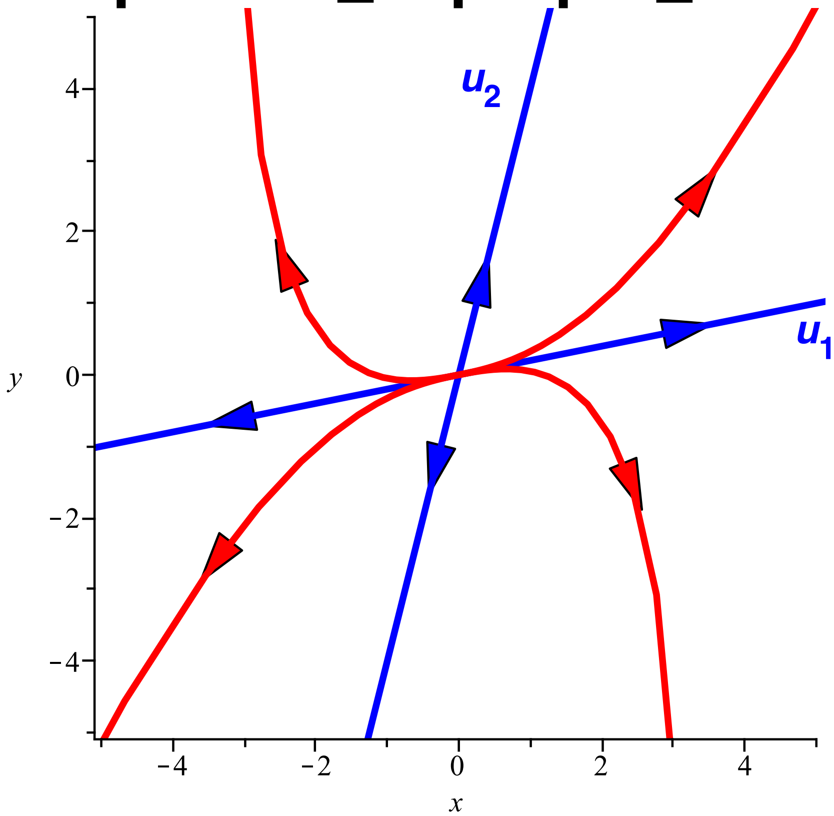
- Prepare an eigenvector $\mathbf{u}_1$ for λ_1: that is, $(A - \lambda_1 I)\mathbf{u}_1 = 0, \mathbf{u}_1 \neq 0$.
- Prepare an eigenvector $\mathbf{u}_2$ for λ_2: that is, $(A - \lambda_2 I)\mathbf{u}_2 = 0, \mathbf{u}_2 \neq 0$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{u}_1 + C_2 e^{\lambda_2 t}\mathbf{u}_2,$$

where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{11}\begin{bmatrix} -35 & 8 \\ -6 & -9 \end{bmatrix}$, $\lambda_1 = -3, \mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\lambda_2 = -1, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Repulsive_improper_node



The eigenvalues of A are: $\lambda_2 > \lambda_1 > 0$.

Solution Method:

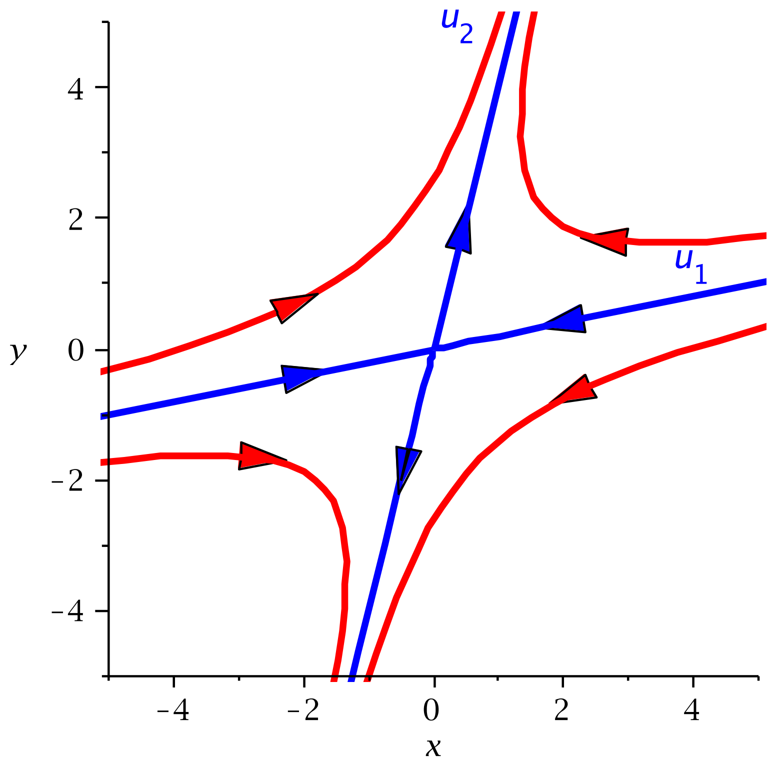
- Prepare an eigenvector $\mathbf{u}_1$ for λ_1: that is, $(A - \lambda_1 I)\mathbf{u}_1 = 0, \mathbf{u}_1 \neq 0$.
- Prepare an eigenvector $\mathbf{u}_2$ for λ_2: that is, $(A - \lambda_2 I)\mathbf{u}_2 = 0, \mathbf{u}_2 \neq 0$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{u}_1 + C_2 e^{\lambda_2 t}\mathbf{u}_2,$$

where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{19}\begin{bmatrix} 35 & 15 \\ -12 & 98 \end{bmatrix}$, $\quad \lambda_1 = 2, \mathbf{u}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\quad \lambda_2 = 5, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Saddle



The eigenvalues of A are: $\lambda_1 < 0 < \lambda_2$.

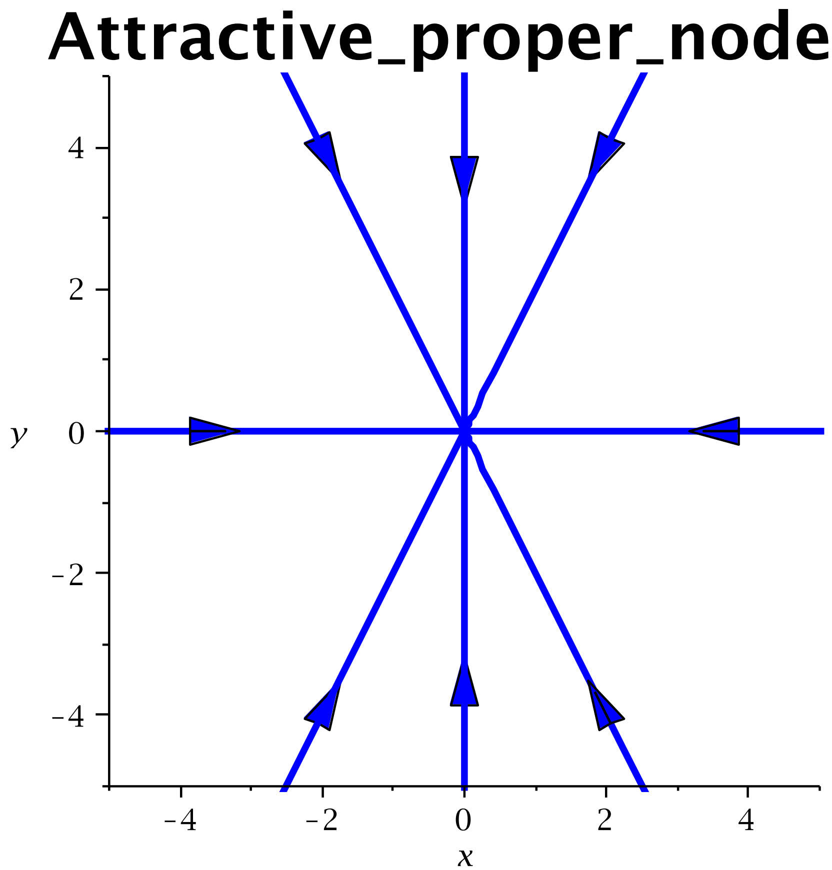
Solution Method:

- Prepare an eigenvector $\mathbf{u}_1$ for λ_1: that is, $(A - \lambda_1 I)\mathbf{u}_1 = 0, \mathbf{u}_1 \neq 0$.
- Prepare an eigenvector $\mathbf{u}_2$ for λ_2: that is, $(A - \lambda_2 I)\mathbf{u}_2 = 0, \mathbf{u}_2 \neq 0$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{u}_1 + C_2 e^{\lambda_2 t}\mathbf{u}_2,$$

where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{19}\begin{bmatrix} -62 & 25 \\ -20 & 43 \end{bmatrix}$, $\lambda_1 = -3, \mathbf{u}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\lambda_2 = 2, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

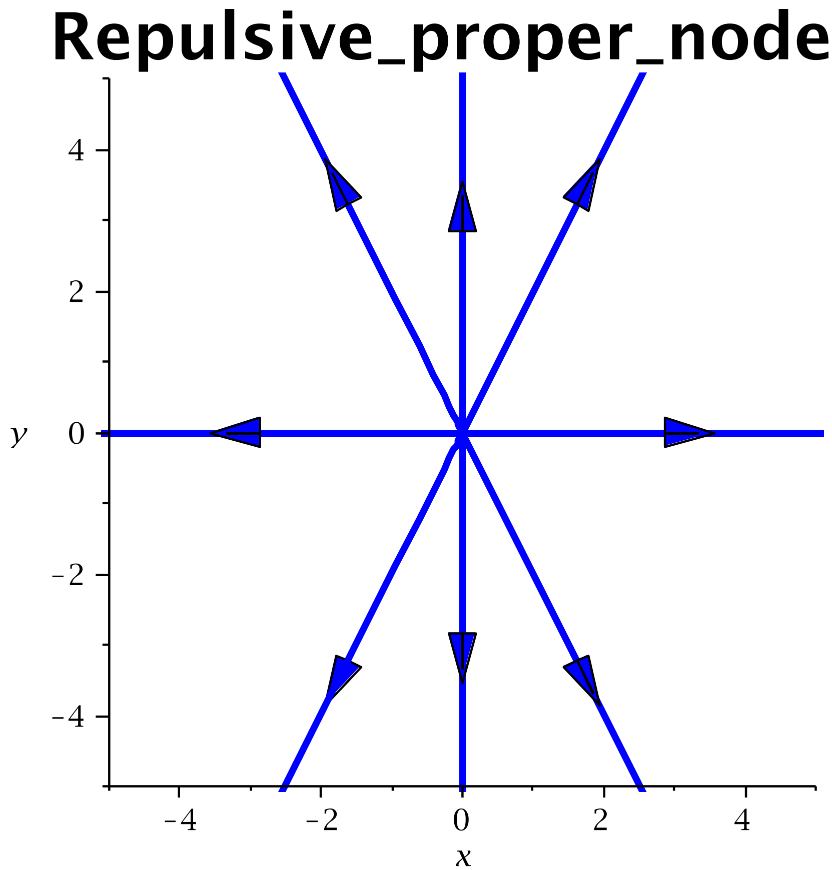


$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda < 0$.

The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

where C_1 and C_2 are free parameters.



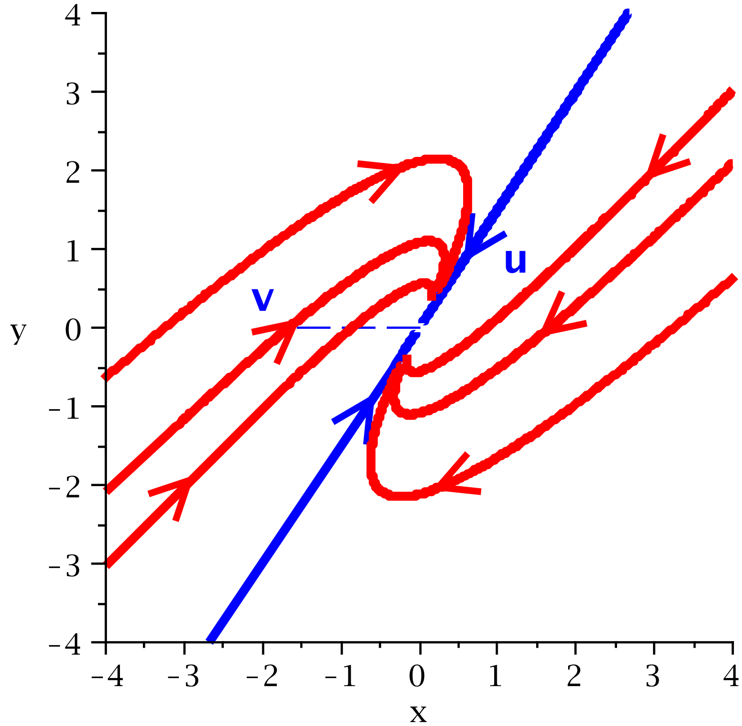
$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda > 0$.

The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

where C_1 and C_2 are free parameters.

Attractive_degenerate_node



The eigenvalues of A are: $\lambda_1 = \lambda_2 < 0$, but $A \neq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$.

Solution Method:

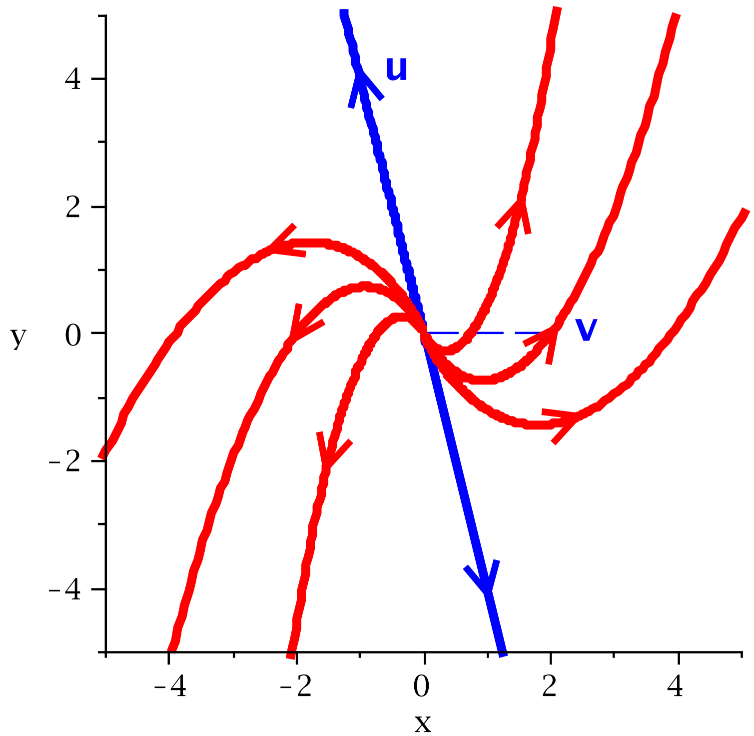
- Prepare an eigenvector $\mathbf{u}$ for λ_1: that is, $(A - \lambda_1 I)\mathbf{u} = 0, \mathbf{u} \neq 0$.
- Prepare a vector $\mathbf{v}$ satisfying $(A - \lambda_1 I)\mathbf{v} = \mathbf{u}$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{u} + C_2 e^{\lambda_1 t}(\mathbf{v} + t\mathbf{u}),$$

where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{15}\begin{bmatrix} -11 & 4 \\ -9 & 1 \end{bmatrix}$, $\quad \lambda_1 = \lambda_2 = -1/3, \mathbf{u} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$, $\quad \mathbf{v} = \begin{bmatrix} -5/3 \\ 0 \end{bmatrix}$.

Repulsive_degenerate_node



The eigenvalues of A are: $\lambda_1 = \lambda_2 > 0$, but $A \neq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$.

Solution Method:

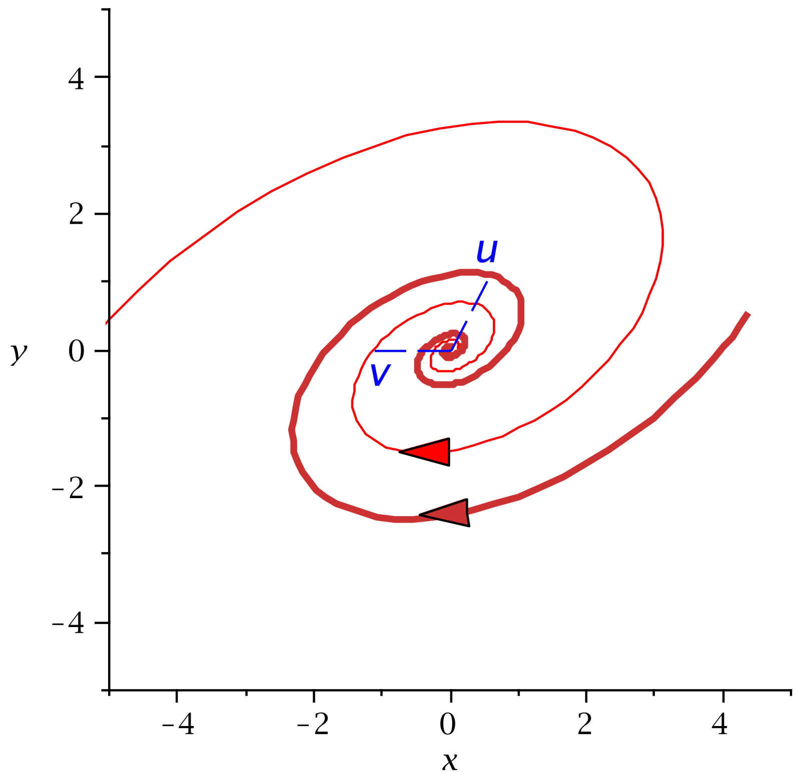
- Prepare an eigenvector $\mathbf{u}$ for λ_1: that is, $(A - \lambda_1 I)\mathbf{u} = 0, \mathbf{u} \neq 0$.
- Prepare a vector $\mathbf{v}$ satisfying $(A - \lambda_1 I)\mathbf{v} = \mathbf{u}$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{u} + C_2 e^{\lambda_1 t}(\mathbf{v} + t\mathbf{u}),$$

where C_1 and C_2 are free parameters.

Example. $A = \begin{bmatrix} 3/2 & -1/8 \\ 2 & 5/2 \end{bmatrix}$, $\lambda_1 = \lambda_2 = 2, \mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Attractive_focus



The eigenvalues of A are: $\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha < 0, \beta > 0$.

Solution Method:

- Prepare an eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ for $\lambda_1 = \alpha + \beta i$: that is, $[A - (\alpha + \beta i)I]\mathbf{w} = 0, \mathbf{w} \neq 0$. Here, $\mathbf{u}$ and $\mathbf{v}$ are respectively the real and imaginary parts of $\mathbf{w}$.

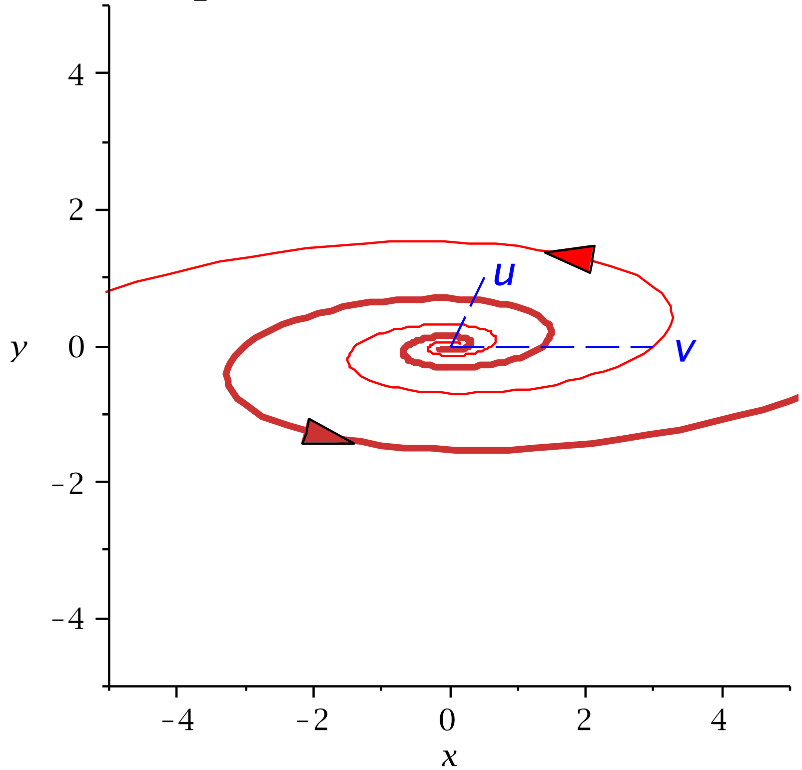
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] + C_2 e^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right],$$

  where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{19}\begin{bmatrix} -55 & 104 \\ -68 & 17 \end{bmatrix}$, $\quad \lambda_1 = -1 + 4i, \mathbf{w} = \begin{bmatrix} 9/17 \\ 1 \end{bmatrix} + i\begin{bmatrix} -19/17 \\ 0 \end{bmatrix}$.

Repulsive_focus



The eigenvalues of A are: $\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha > 0, \beta > 0$.

Solution Method:

- Prepare an eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ for $\lambda_1 = \alpha + \beta i$: that is, $[A - (\alpha + \beta i)I]\mathbf{w} = 0, \mathbf{w} \neq 0$. Here, $\mathbf{u}$ and $\mathbf{v}$ are respectively the real and imaginary parts of $\mathbf{w}$.

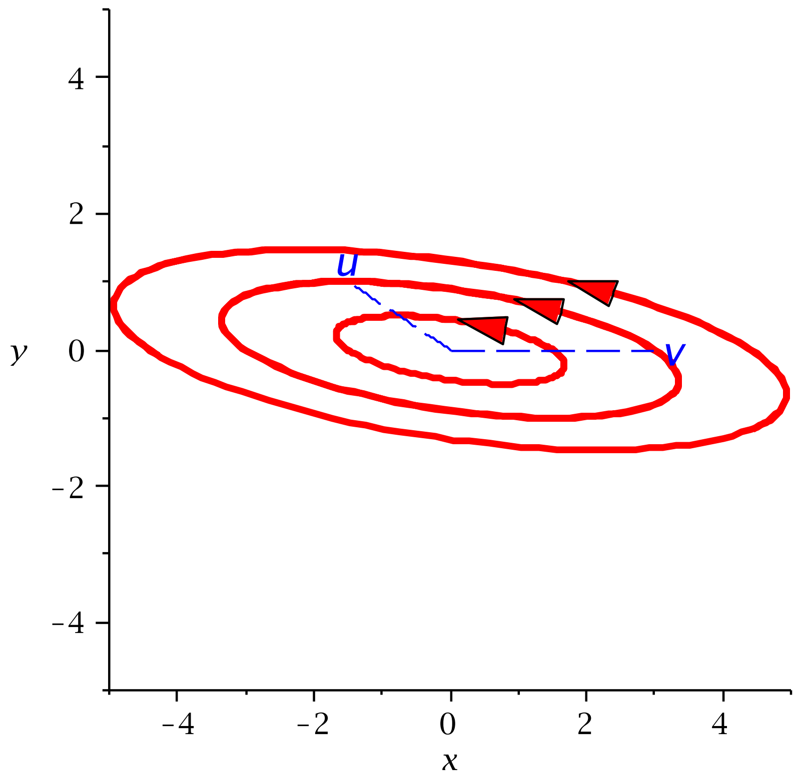
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\alpha t} \left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] + C_2 e^{\alpha t} \left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right],$$

  where C_1 and C_2 are free parameters.

Example. $A = \begin{bmatrix} 5/2 & -37/4 \\ 1 & 3/2 \end{bmatrix}$, $\lambda_1 = 2 + 3i, \mathbf{w} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Center



The eigenvalues of A are: $\lambda_{1,2} = \pm \beta i$ with $\beta > 0$.

Solution Method:

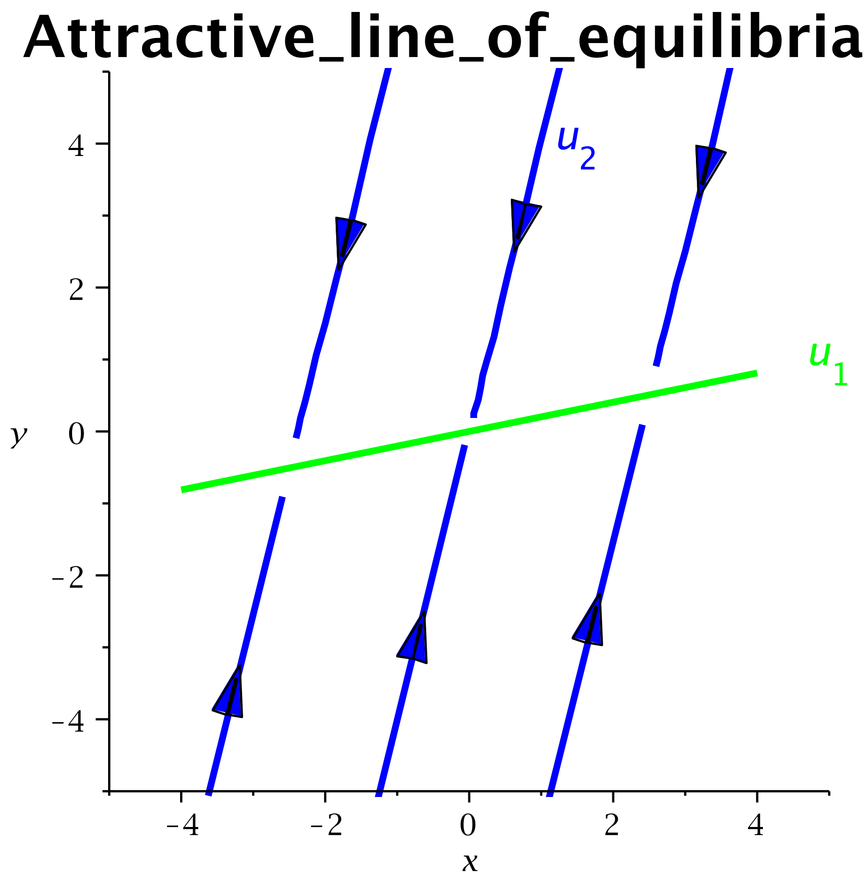
- Prepare an eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ for $\lambda_1 = \beta i$: that is, $(A - \beta i\, I)\mathbf{w} = 0, \mathbf{w} \neq 0$. Here, $\mathbf{u}$ and $\mathbf{v}$ are respectively the real and imaginary parts of $\mathbf{w}$.

- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right] + C_2 \left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right],$$

  where C_1 and C_2 are free parameters.

Example. $A = \begin{bmatrix} -2 & -15 \\ 4/3 & 2 \end{bmatrix}$, $\quad \lambda_1 = 4i, \mathbf{w} = \begin{bmatrix} -\frac{3}{2} + 3i \\ 1 \end{bmatrix}$ with $\beta = 4, \mathbf{u} = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.



The eigenvalues of A are: $\lambda_1 = 0, \lambda_2 < 0$.

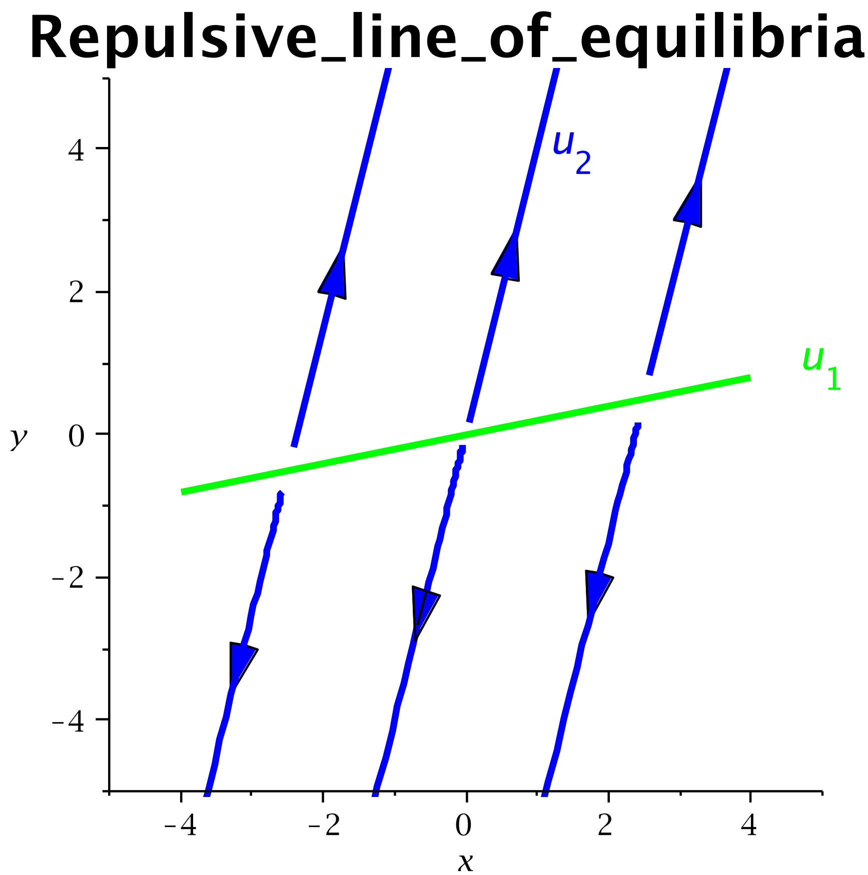
Solution Method:

- Prepare an eigenvector $\mathbf{u}_1$ for $\lambda_1 = 0$: that is, $A\mathbf{u}_1 = 0, \mathbf{u}_1 \neq 0$.
- Prepare an eigenvector $\mathbf{u}_2$ for λ_2: that is, $(A - \lambda_2 I)\mathbf{u}_2 = 0, \mathbf{u}_2 \neq 0$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\mathbf{u}_1 + C_2 e^{\lambda_2 t}\mathbf{u}_2,$$

  where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{19}\begin{bmatrix} 2 & -10 \\ 8 & -40 \end{bmatrix}, \quad \lambda_1 = 0, \mathbf{u}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$



The eigenvalues of A are: $\lambda_1 = 0, \lambda_2 > 0$.

Solution Method:

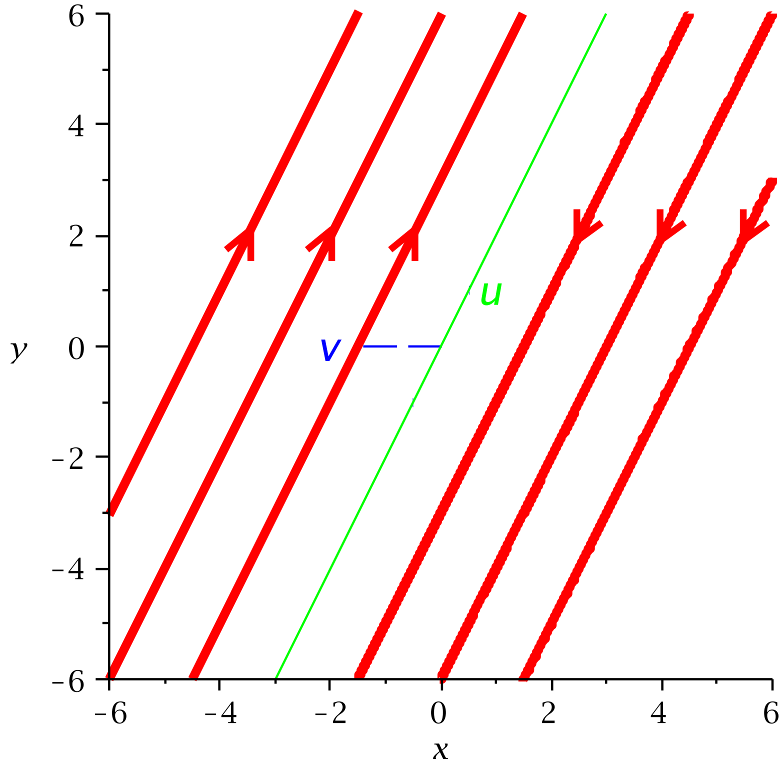
- Prepare an eigenvector $\mathbf{u}_1$ for $\lambda_1 = 0$: that is, $A\mathbf{u}_1 = 0, \mathbf{u}_1 \neq 0$.
- Prepare an eigenvector $\mathbf{u}_2$ for λ_2: that is, $(A - \lambda_2 I)\mathbf{u}_2 = 0, \mathbf{u}_2 \neq 0$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\mathbf{u}_1 + C_2 e^{\lambda_2 t}\mathbf{u}_2,$$

where C_1 and C_2 are free parameters.

Example. $A = \frac{1}{19}\begin{bmatrix} -2 & 10 \\ -8 & 40 \end{bmatrix}, \quad \lambda_1 = 0, \mathbf{u}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$

Laminated_flow



The eigenvalues of A are: $\lambda_1 = \lambda_2 = 0$, but $A \neq 0$.

Solution Method:

- Prepare an eigenvector $\mathbf{u}$ for $\lambda_1 = 0$: that is, $A\mathbf{u} = 0, \mathbf{u} \neq 0$.
- Prepare a vector $\mathbf{v}$ satisfying $A\mathbf{v} = \mathbf{u}$.
- The general solutions of the differential system are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\mathbf{u} + C_2(\mathbf{v} + t\mathbf{u}),$$

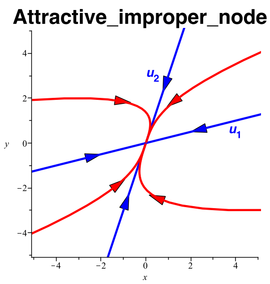
where C_1 and C_2 are free parameters.

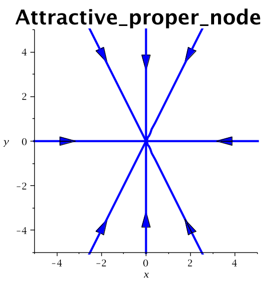
Example. $A = \begin{bmatrix} -1/3 & 1/6 \\ -2/3 & 1/3 \end{bmatrix}$, $\quad \lambda_1 = \lambda_2 = 0, \mathbf{u} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $\quad \mathbf{v} = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$.

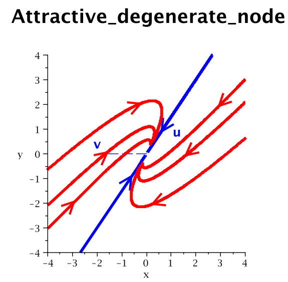
A Topological Classfication of 2-D Linear Flows

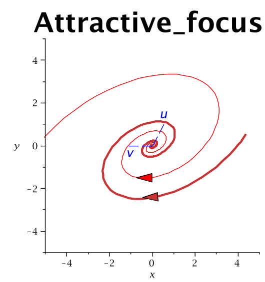
- Attractive nodes and attractice foci.

The following four types are topologically equivalent:





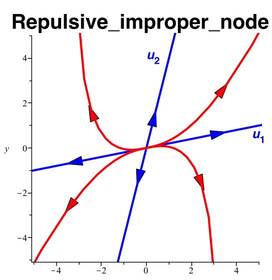


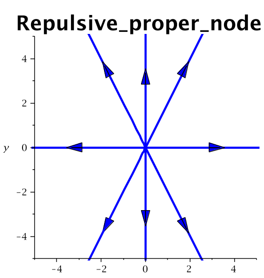


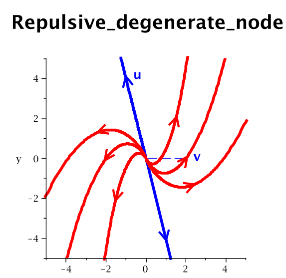
Here, $(x, y) = (0, 0)$ is an asymptotically stable equilibrium.

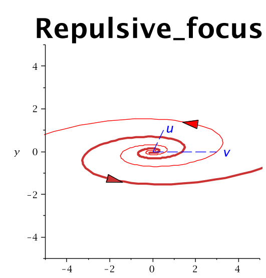
- Repulsive nodes and repulsive foci.

The following four types are topologically equivalent:



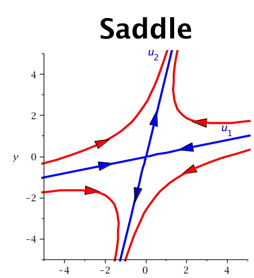






Here, $(x, y) = (0, 0)$ is an unstable equilibrium.

- Saddle.

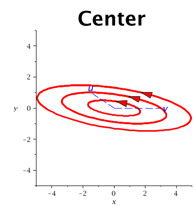


Here, $(x, y) = (0, 0)$ is an unstable equilibrium.

Remark: In the above cases, matrix A has neither zero eigenvalue nor purely imaginary eigenvalues. These flows are *structurally stable* in the sense that they are robust under small perturbations. A small perturbation of the flow will not change the topological type of the phase portrait.

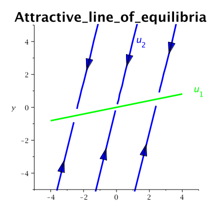
Remark: In the following cases, matrix A has either a zero eigenvalue or purely imaginary eigenvalues. These flows are *structurally unstable* in the sense that they are sensitive to perturbations. An appropriate small perturbation of the flow will completely change the topological type of the phase portrait.

- Center.



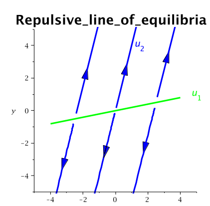
Here, $(x, y) = (0, 0)$ is a stable equilibrium, but is not asymptotically stable.

- Attractive line of equilibria.



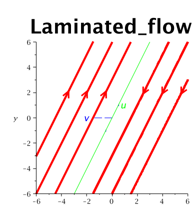
Here, $(x, y) = (0, 0)$ is a stable equilibrium, but is not asymptotically stable.

- Repulsive line of equilibria.



Here, $(x, y) = (0, 0)$ is an unstable equilibrium.

- Laminated flow.



Here, $(x, y) = (0, 0)$ is an unstable equilibrium.

- The whole plane consists of equilibria. That is the case of $A = 0$, where every point on the plane is a constant solution.