Second Order Homogeneous Linear ODEs with Constant Coefficients

Xu-Yan Chen

Diff Eqs: $ay'' + by' + cy = 0$

($a \neq 0$ and a, b, c are real constants)

Things to explore:

- General solutions
- Initial value problems
- Graph solutions y vs t
- Phase portraits in the (y, y') plane
- Stability/instability of equilibrium $(0, 0)$

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The Solution Method:

- $a\lambda^2 + b\lambda + c = 0$ (characteristic polynomial)
  $\Rightarrow \lambda_1, \lambda_2$ (characteristic roots)

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Root structure	**General solutions**
$\lambda_1 \neq \lambda_2$	$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$	$y = C_1 e^{\lambda_1 t} + C_2\, t\, e^{\lambda_1 t}$
$\lambda_{1,2} = \alpha \pm \beta i\ (\beta > 0)$	$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$

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- The Initial Value Problem $\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_1 \end{cases}$

  Initial condition $\Rightarrow C_1, C_2 \Rightarrow$ Unique solution $y(t)$

2nd Ord Diff Eq **Equivalent System**

$$ay'' + by' + cy = 0 \iff \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

by setting $(x_1, x_2) = (y, y')$

- **Eigenvalues:** $a\lambda^2 + b\lambda + c = 0 \iff \det(A - \lambda I) = 0$
- **Distinct roots:** $\lambda_1 \neq \lambda_2, \quad y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$$\Rightarrow \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + C_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

- **Repeated roots:** $\lambda_1 = \lambda_2, \quad y = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$

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- **Complex roots:** $\lambda_{1,2} = \alpha \pm \beta i, \quad y = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$

$$\Rightarrow \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{\alpha t} \left(\cos(\beta t) \begin{bmatrix} 1 \\ \alpha \end{bmatrix} - \sin(\beta t) \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right) + C_2 e^{\alpha t} \left(\sin(\beta t) \begin{bmatrix} 1 \\ \alpha \end{bmatrix} + \cos(\beta t) \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right)$$

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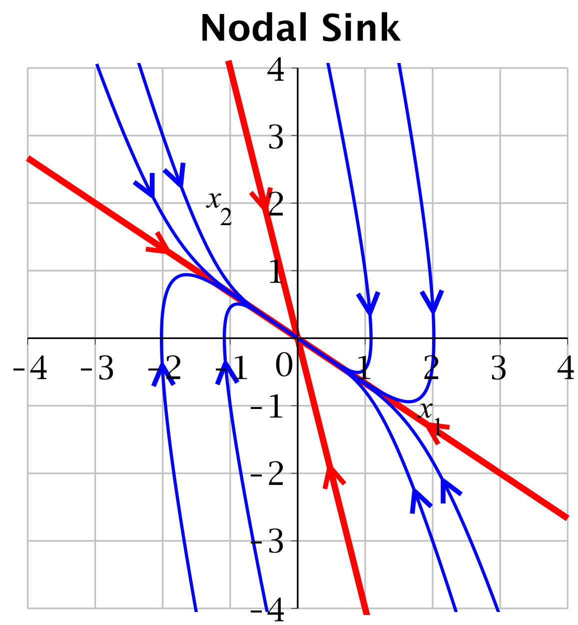

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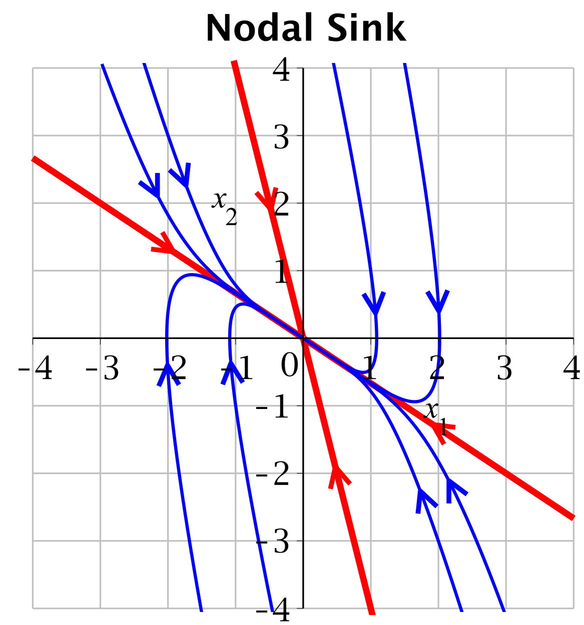

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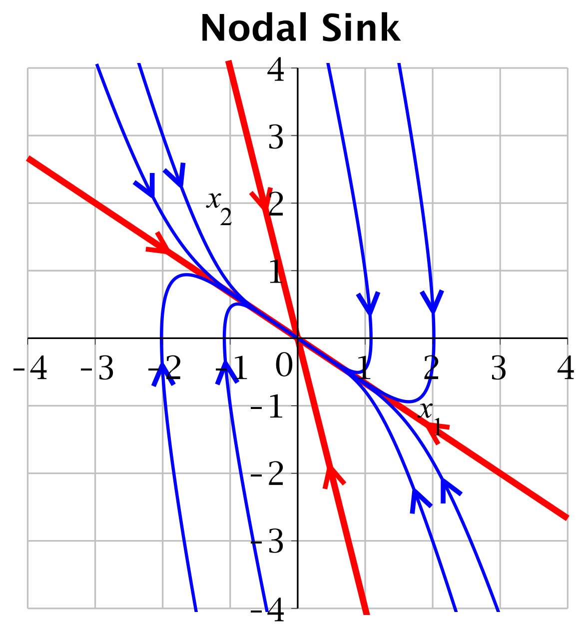

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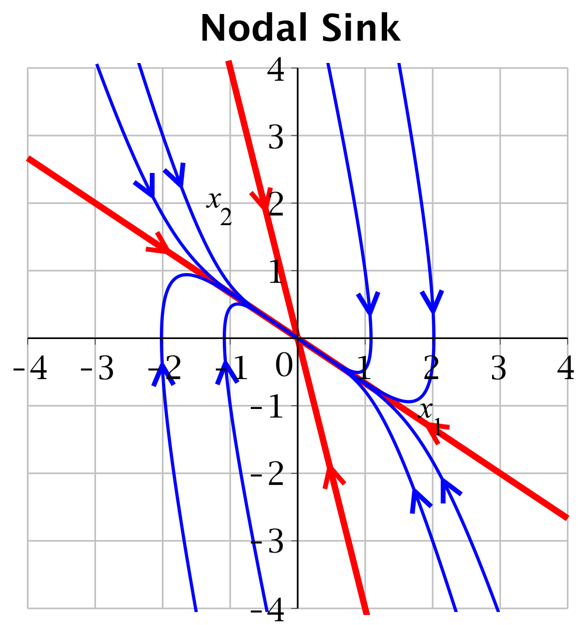

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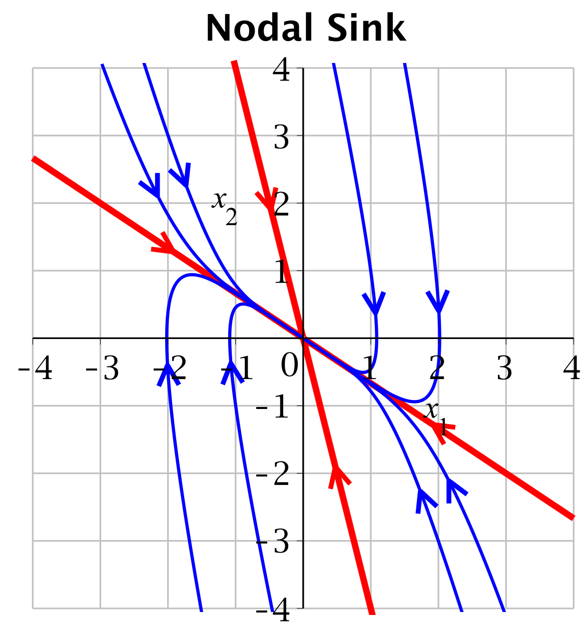

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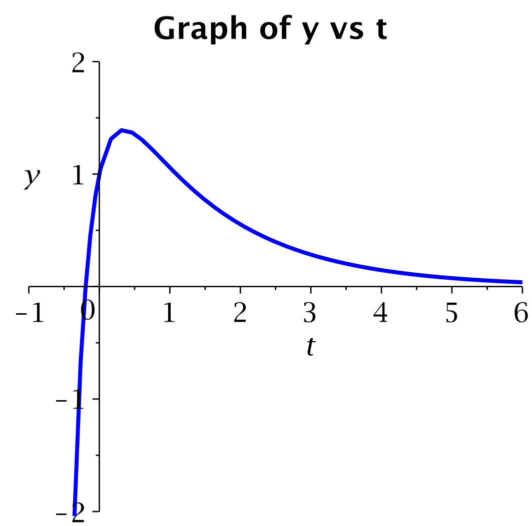

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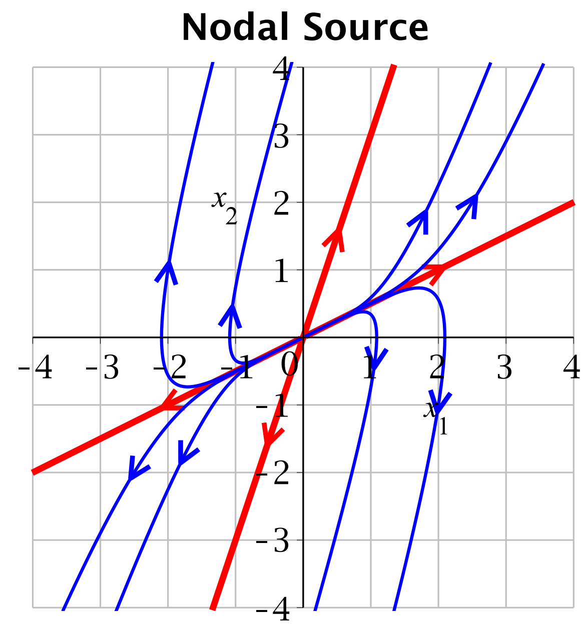

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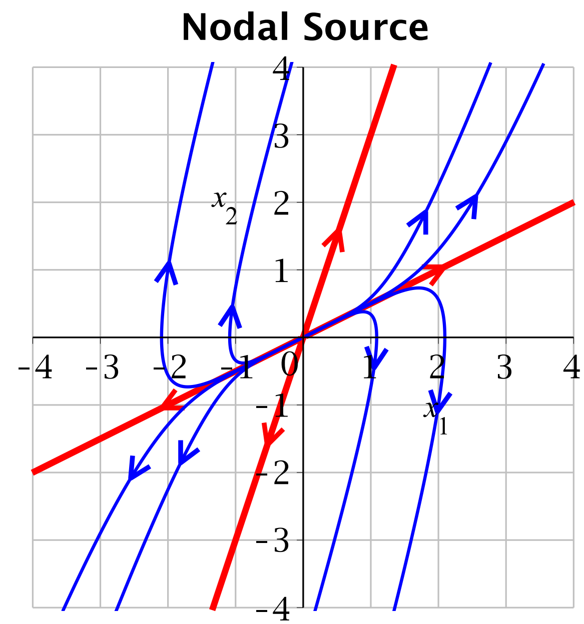

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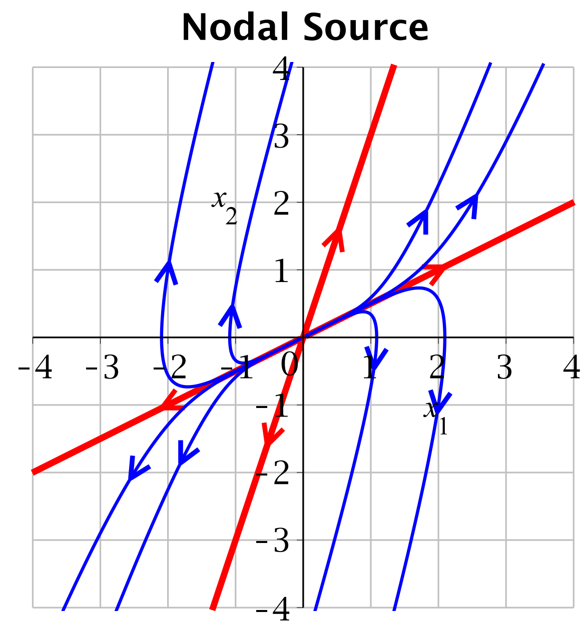

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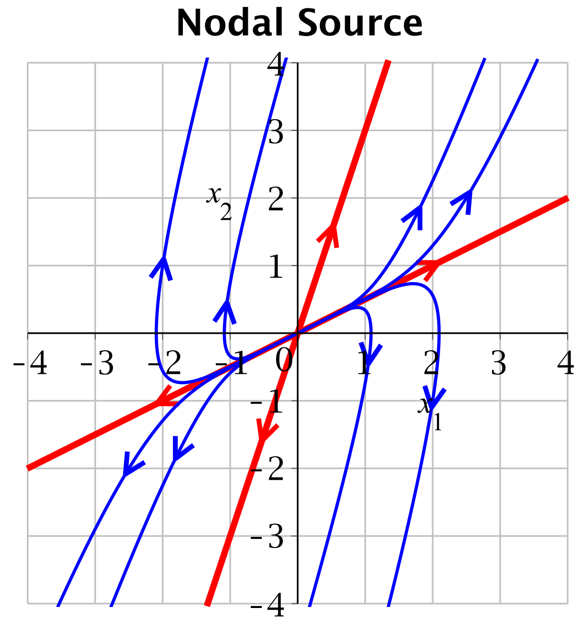

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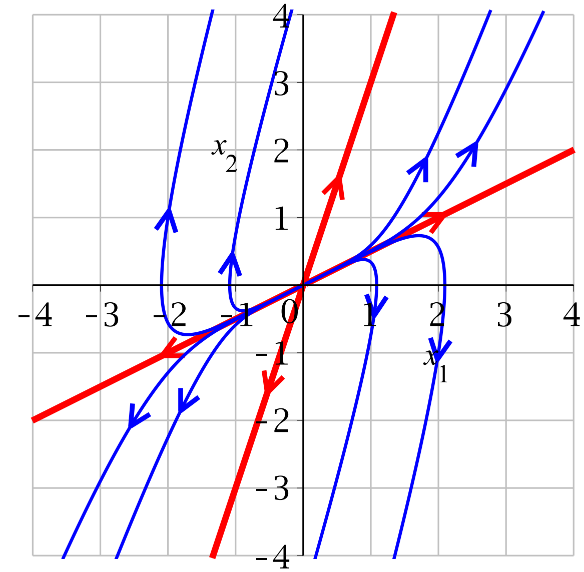
Nodal Source



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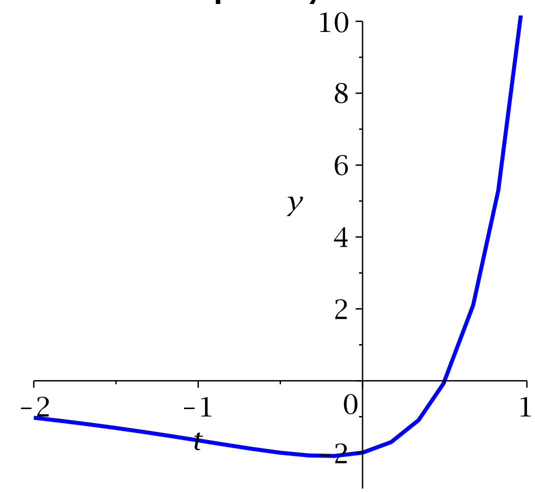
Graph of y vs t



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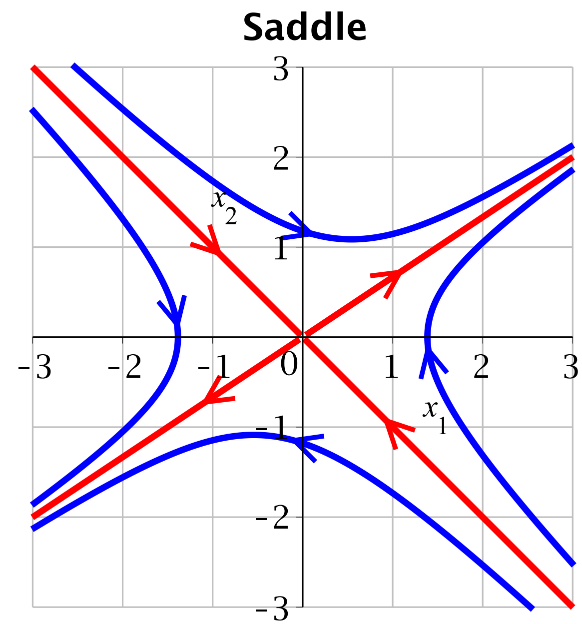

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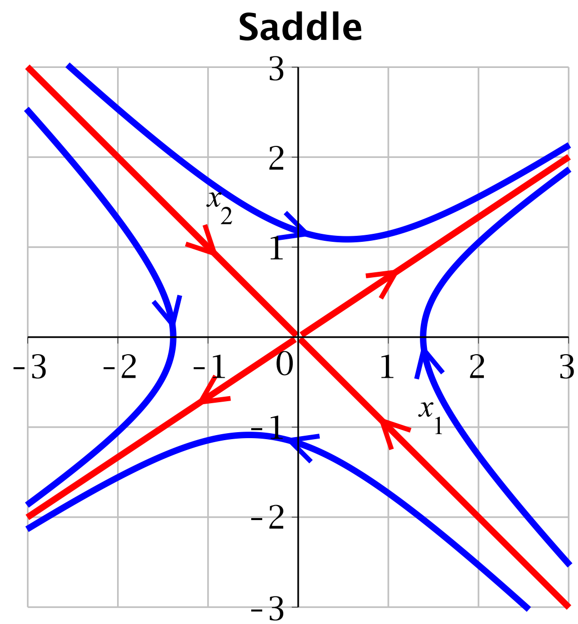

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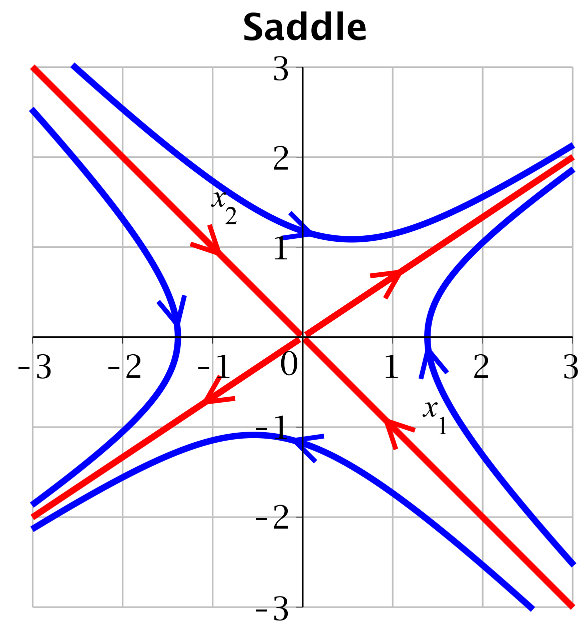

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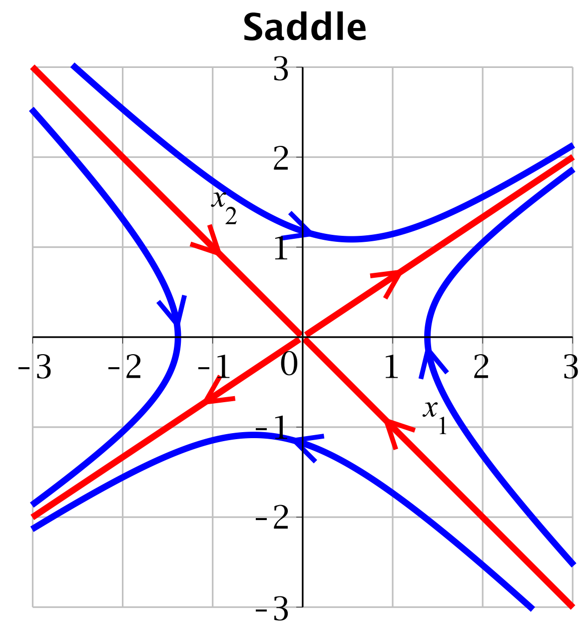

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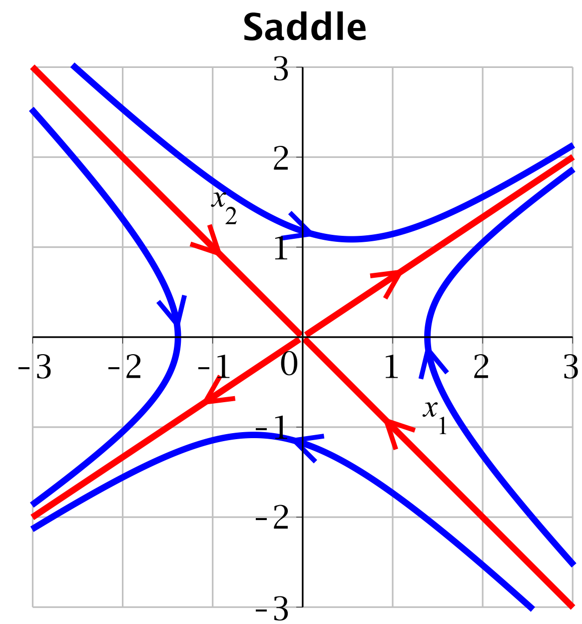

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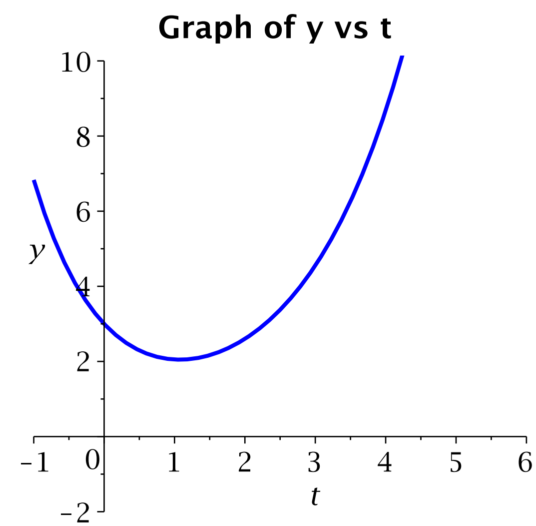
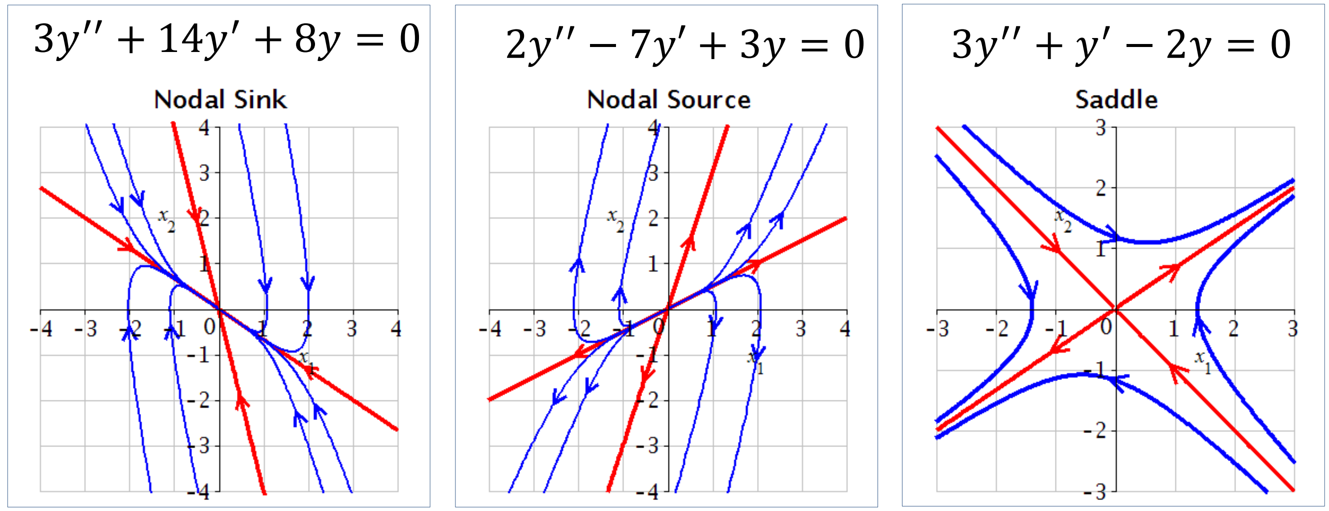


Special Properties of Phase Portraits of 2nd Order Diff Eqs:

- All curves in the upper half plane go to right.
- All curves in the lower half plane go to left.
- The switching occurs at the x_1-intercept, in case a curve goes across the x_1 axis.



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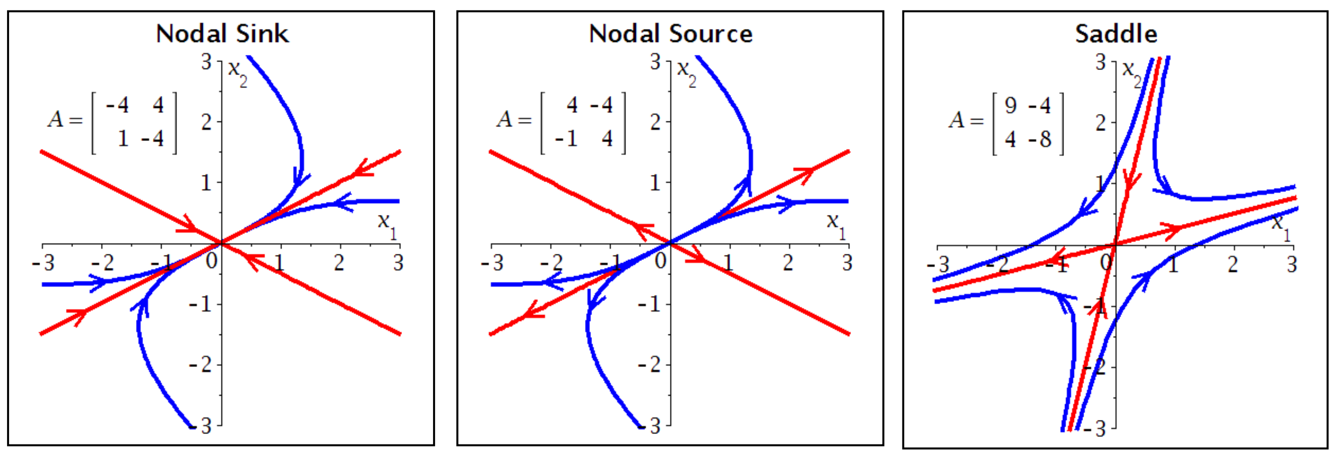

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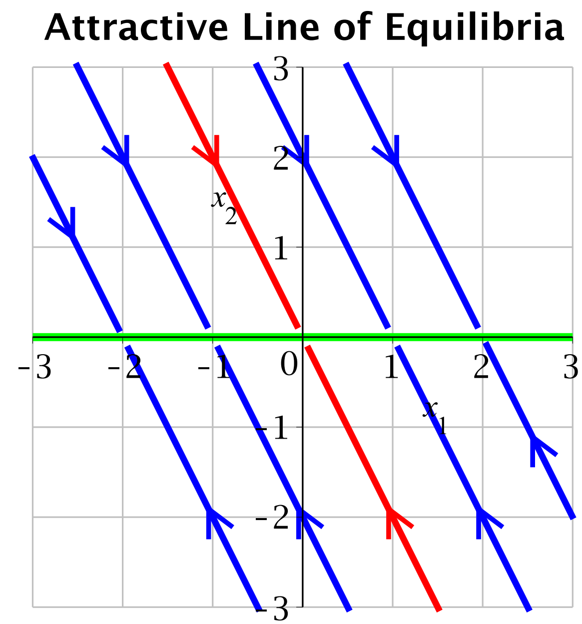

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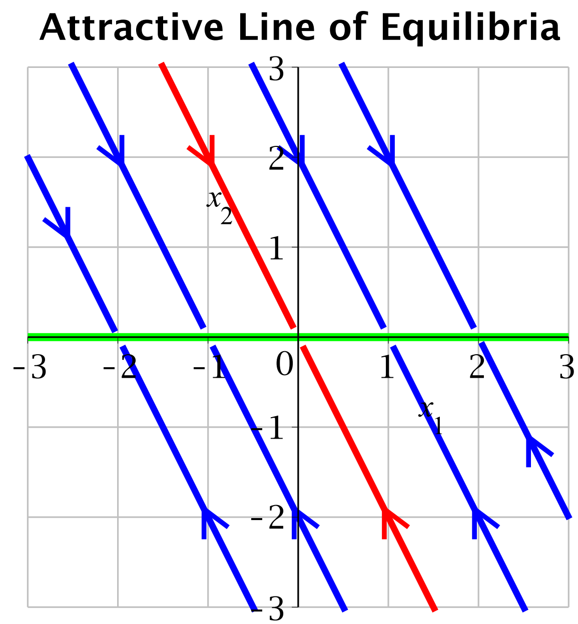

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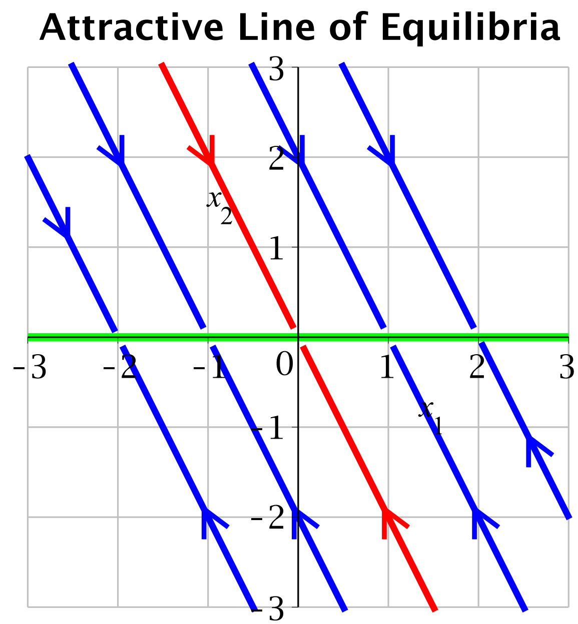

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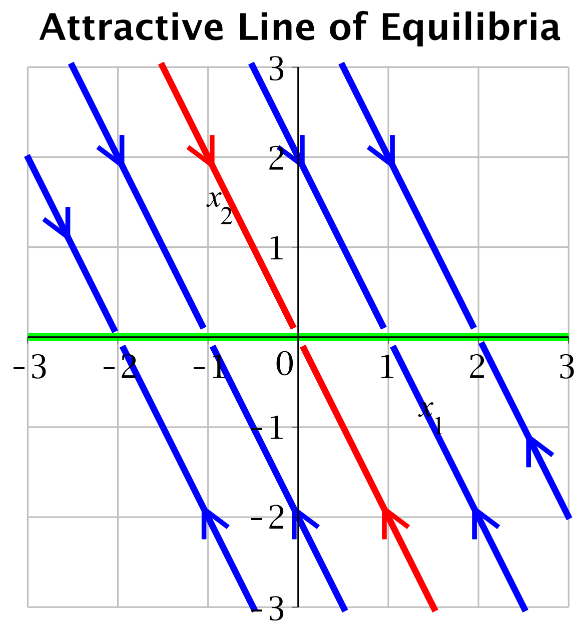

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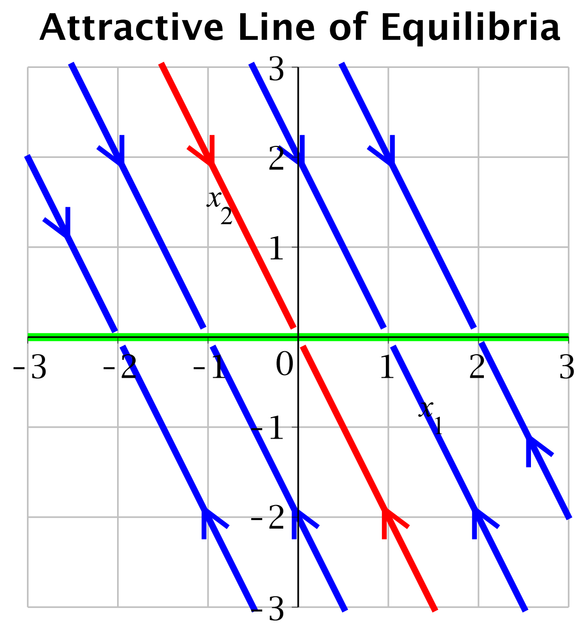

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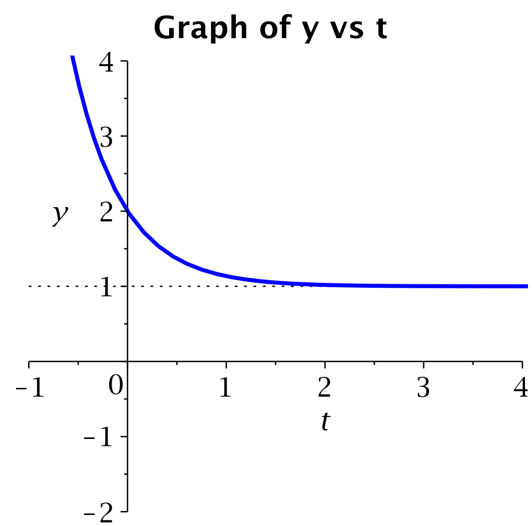

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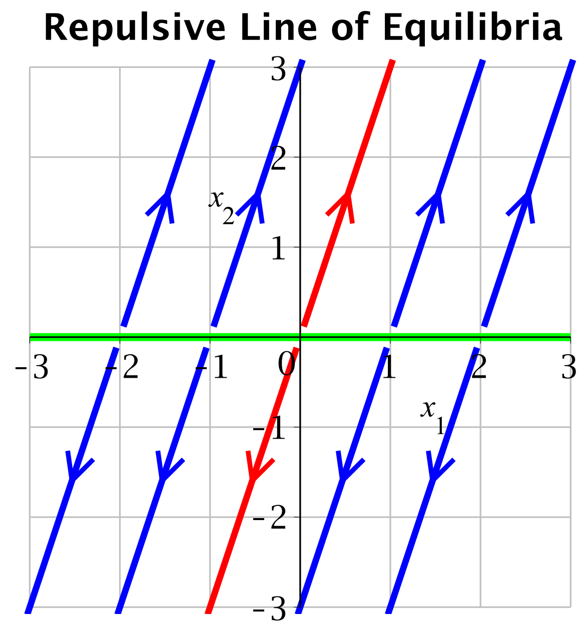

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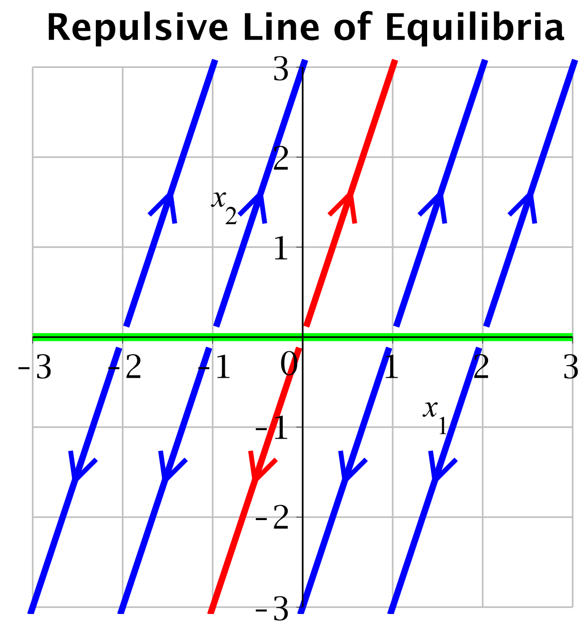

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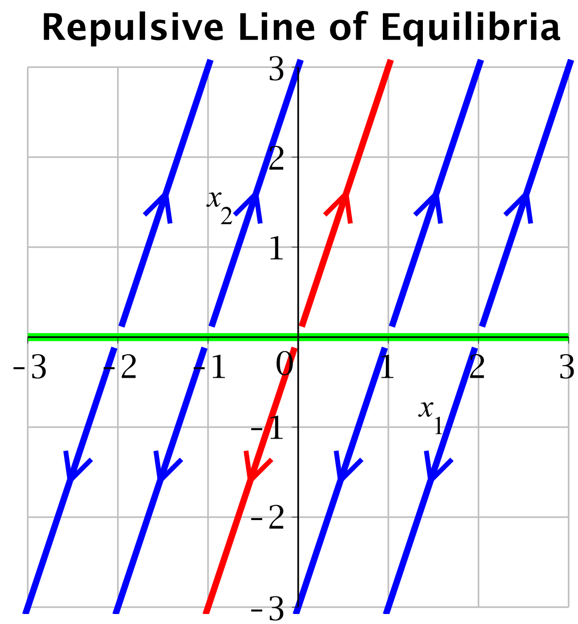

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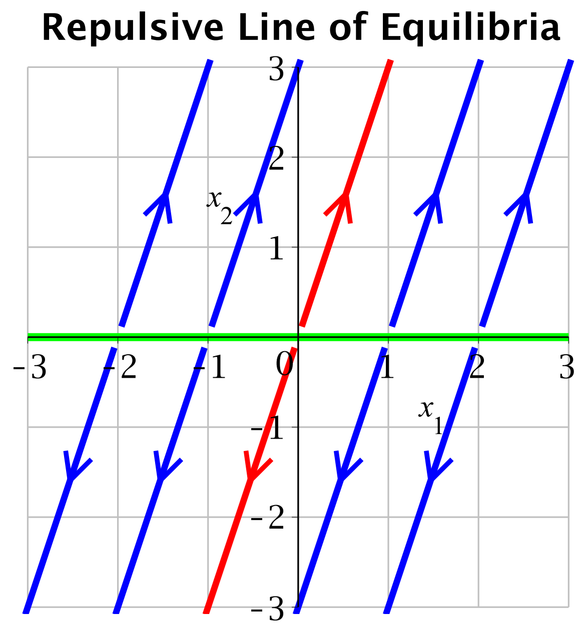

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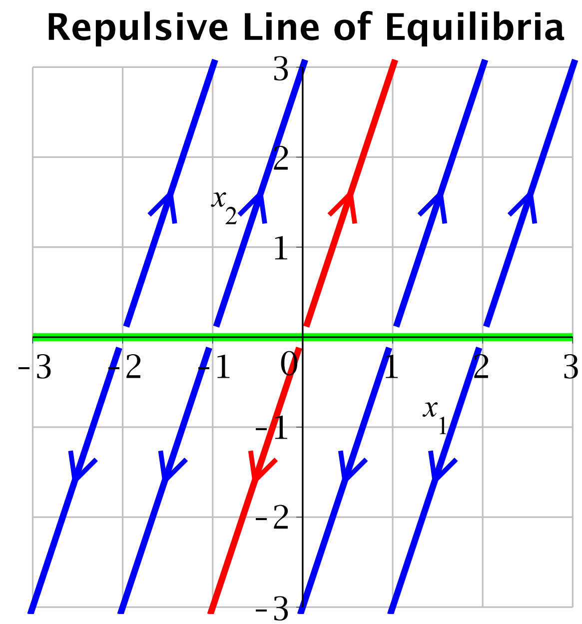

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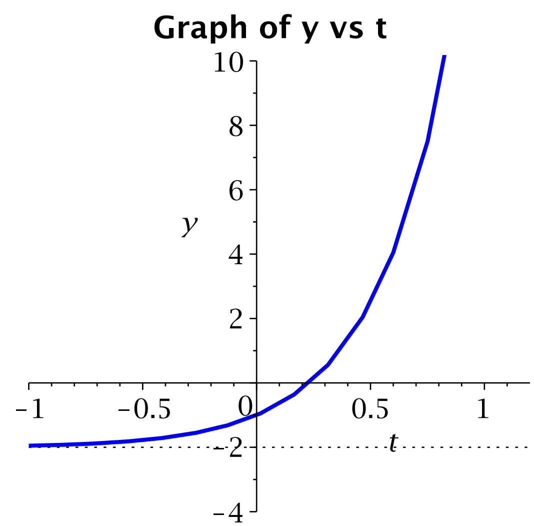

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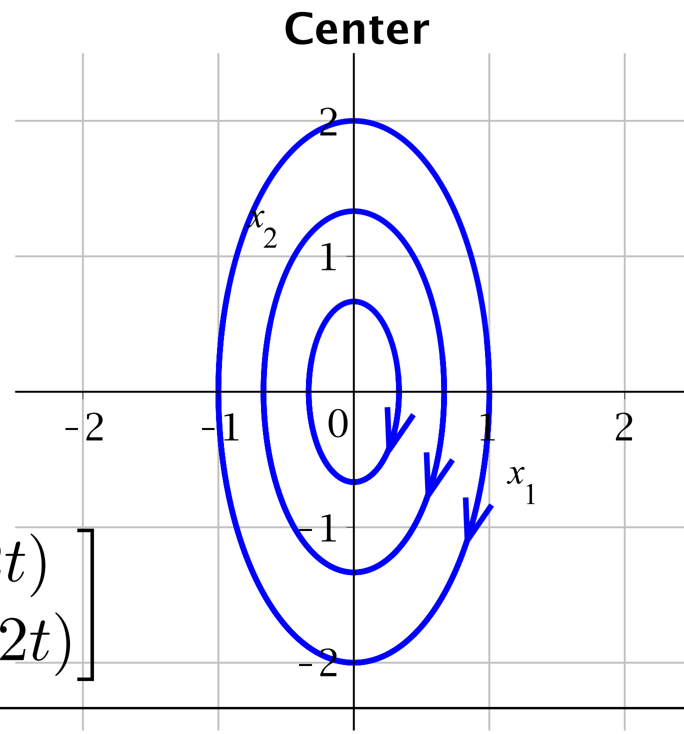

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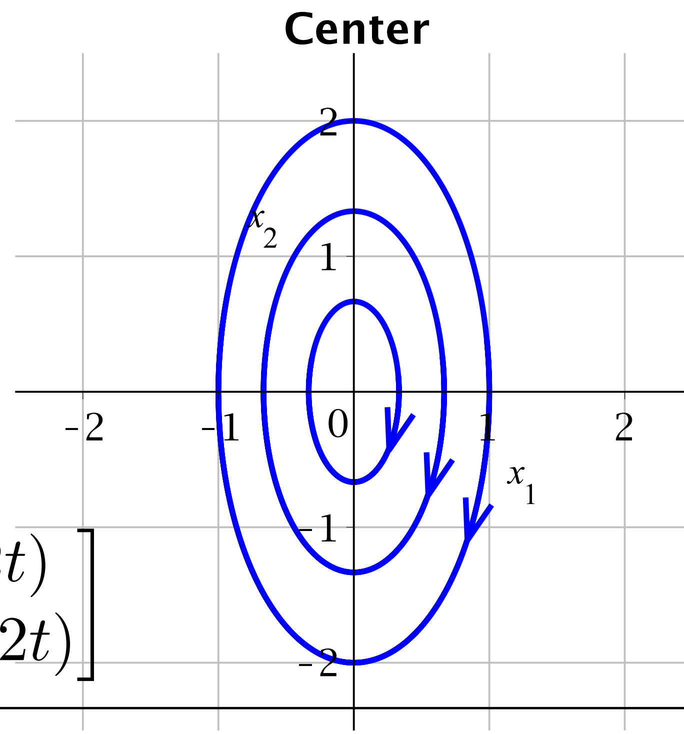

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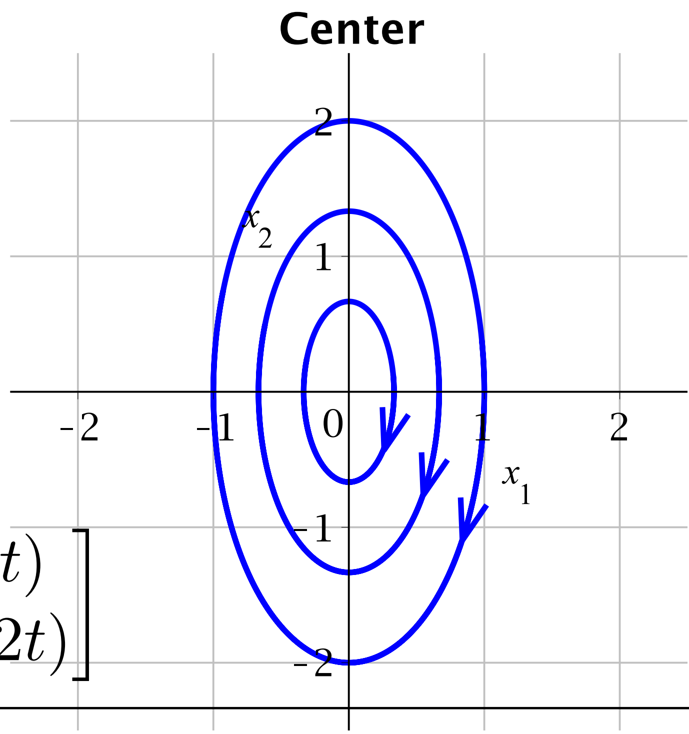

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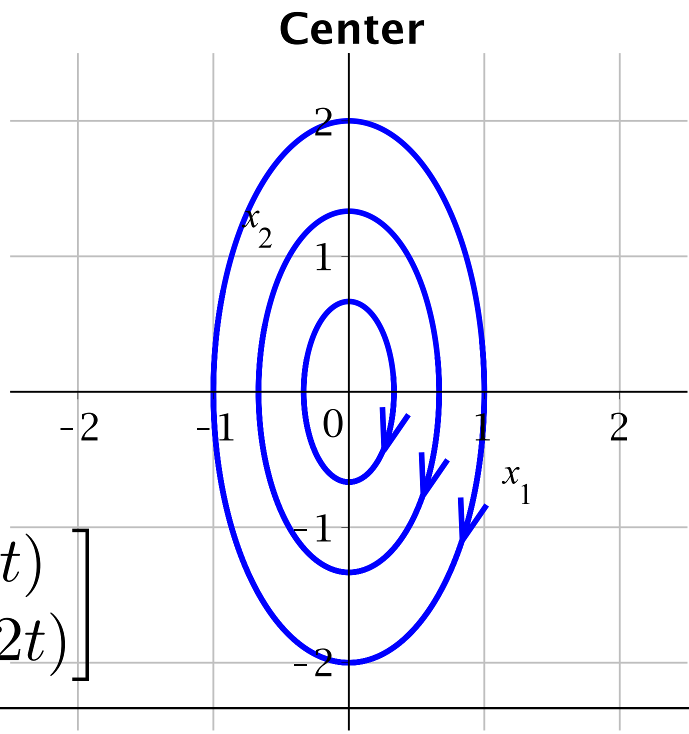

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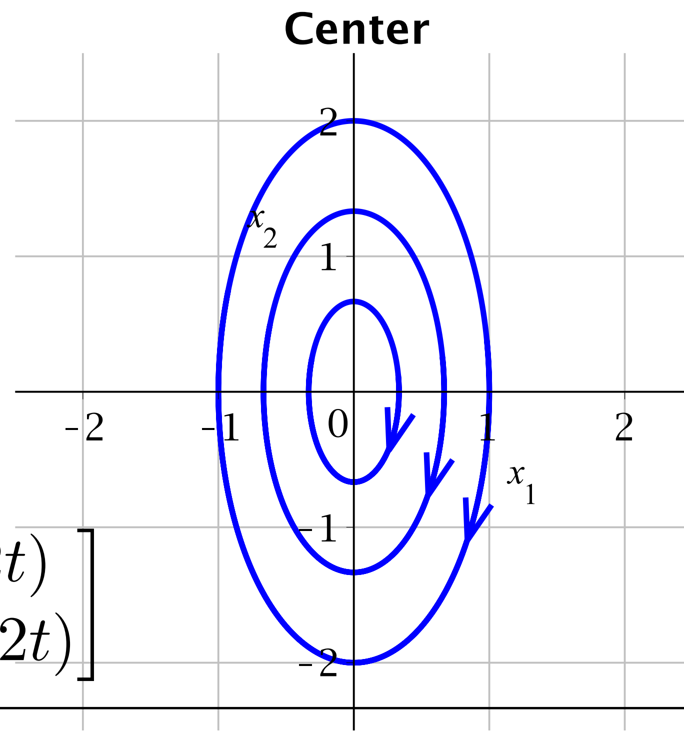

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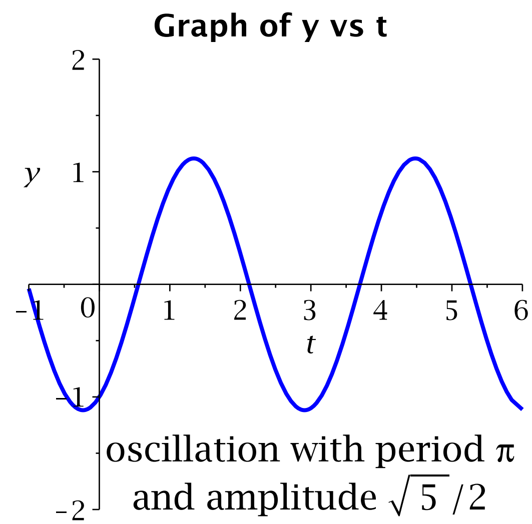
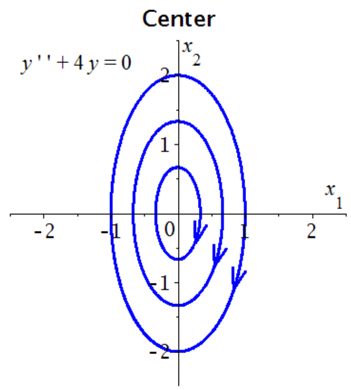
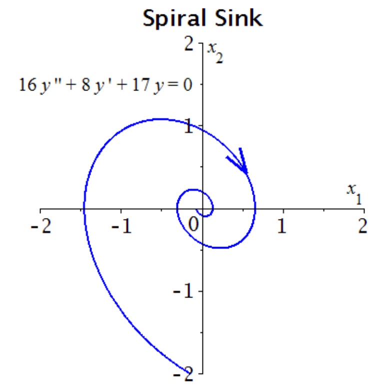
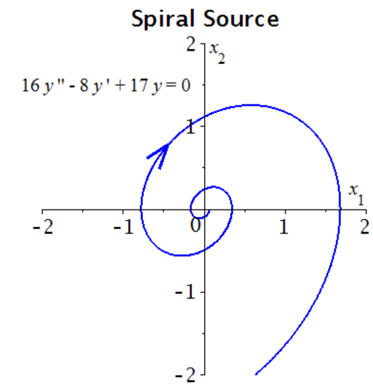


Phase Portrait of $ay'' + by' + cy = 0$ in the Case of Complex Eigenvalues (i.e. $b^2 - 4ac < 0$)

- The solution trajectories always **rotate clockwise.**







Phase Portrait of $ay'' + cy = 0$ in the Case of a Center (i.e. $ac > 0,\ b = 0$)

- The solution curves are ellipses that **rotate clockwise.**
- The symmetric axes are in the x_1 and x_2 directions.
- $\dfrac{\text{The radius in the } x_1 \text{ direction}}{\text{The radius in the } x_2 \text{ direction}} = \sqrt{a:c}$

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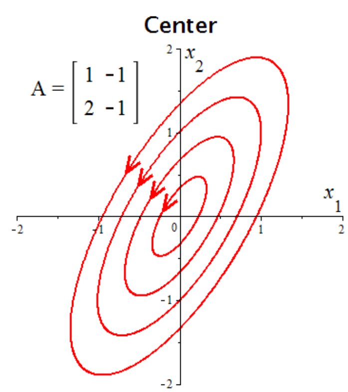
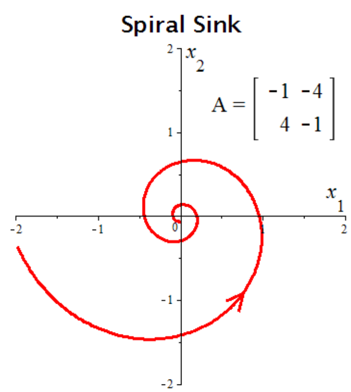
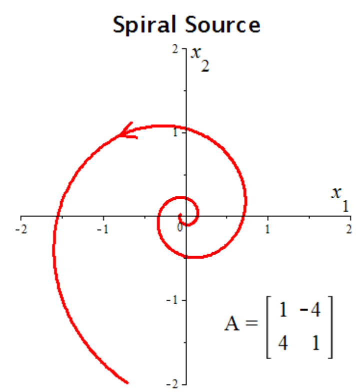
The above are **invalid** for general 2-D systems:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$







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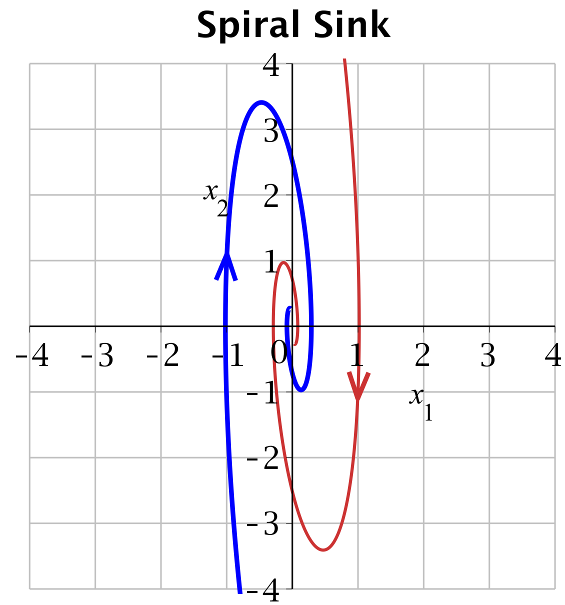

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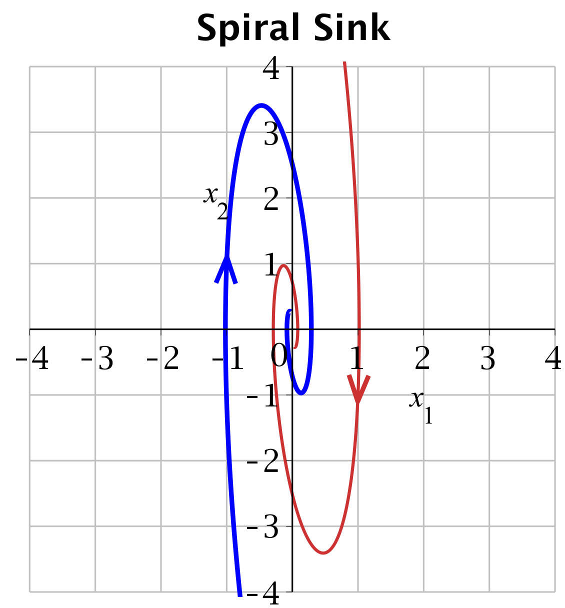

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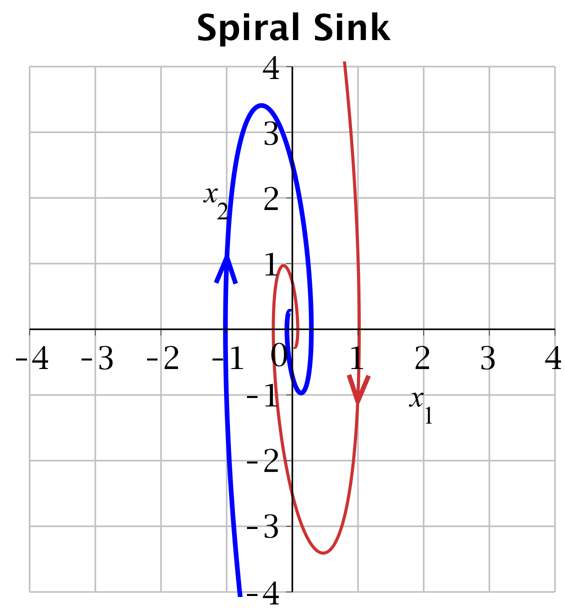

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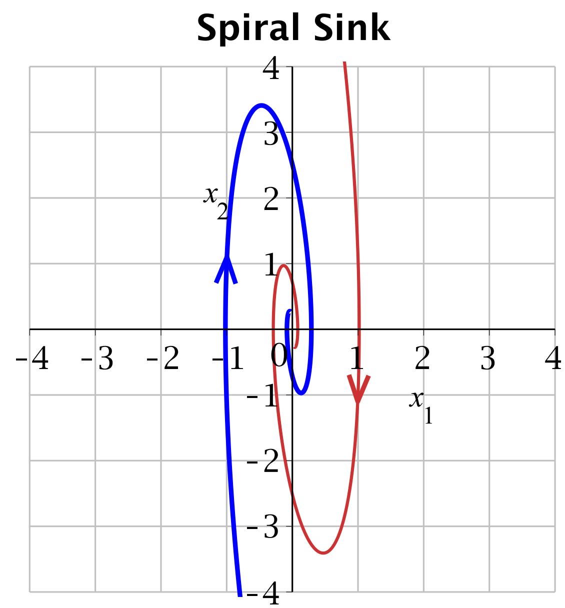

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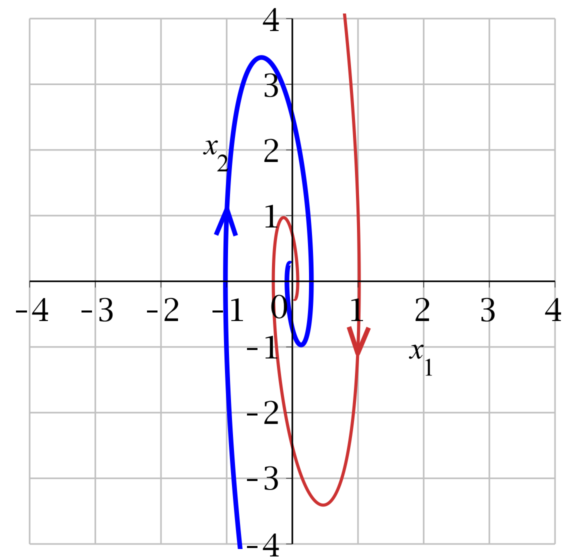

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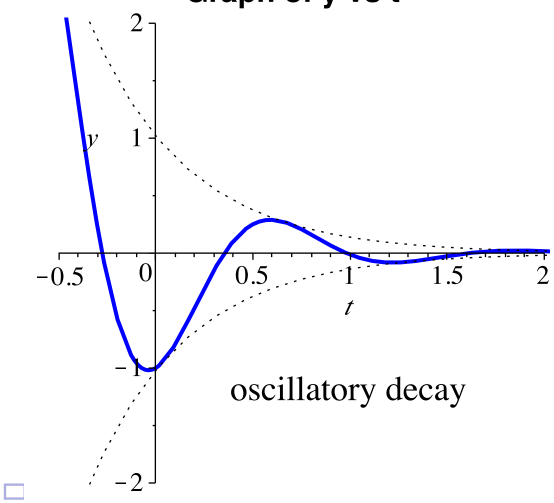

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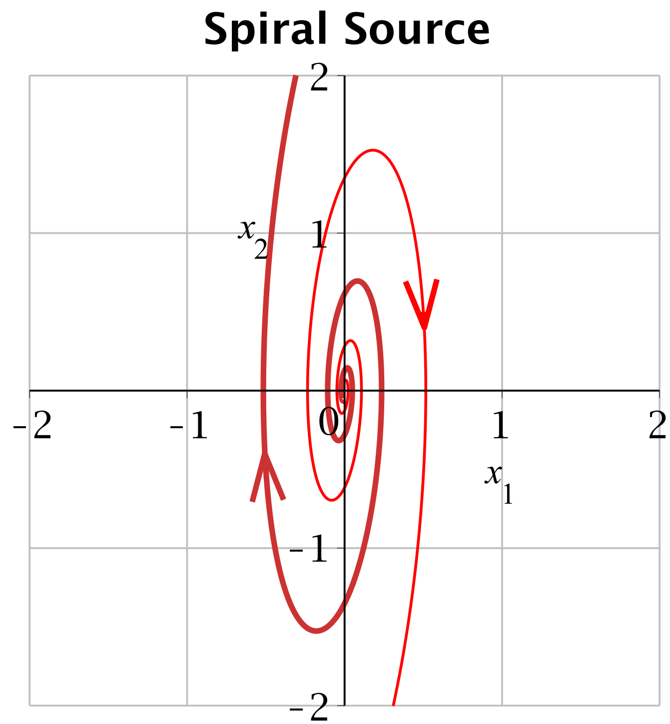

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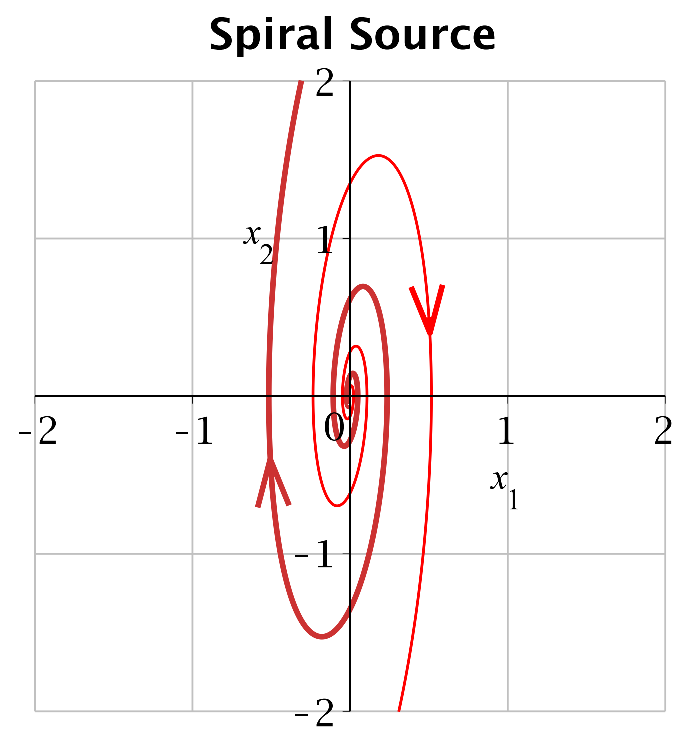

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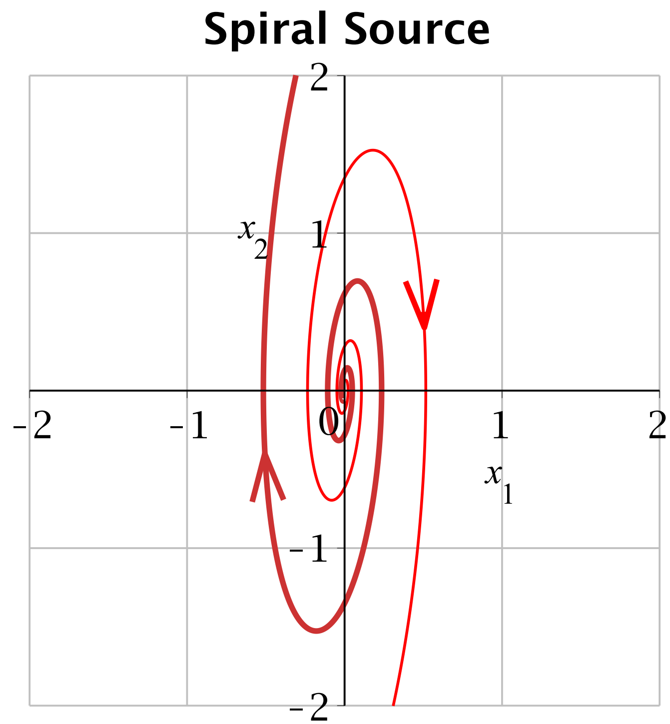

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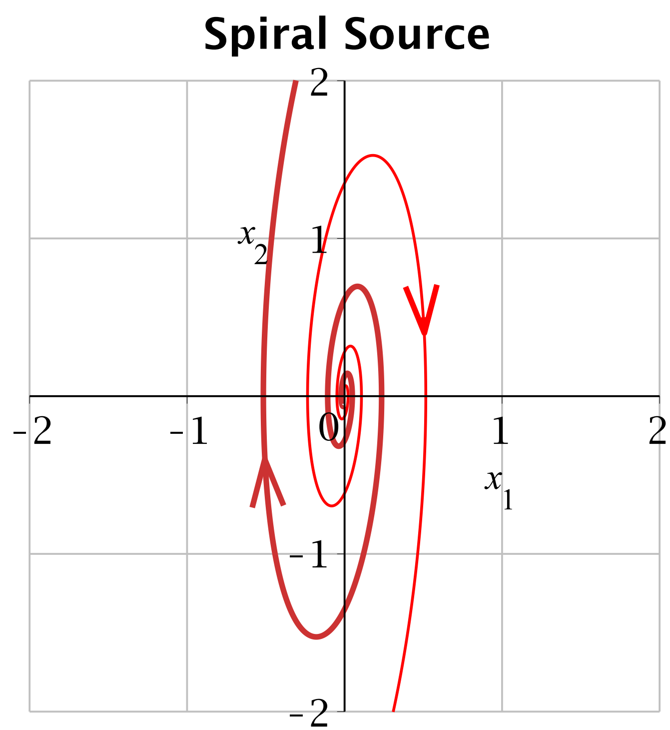

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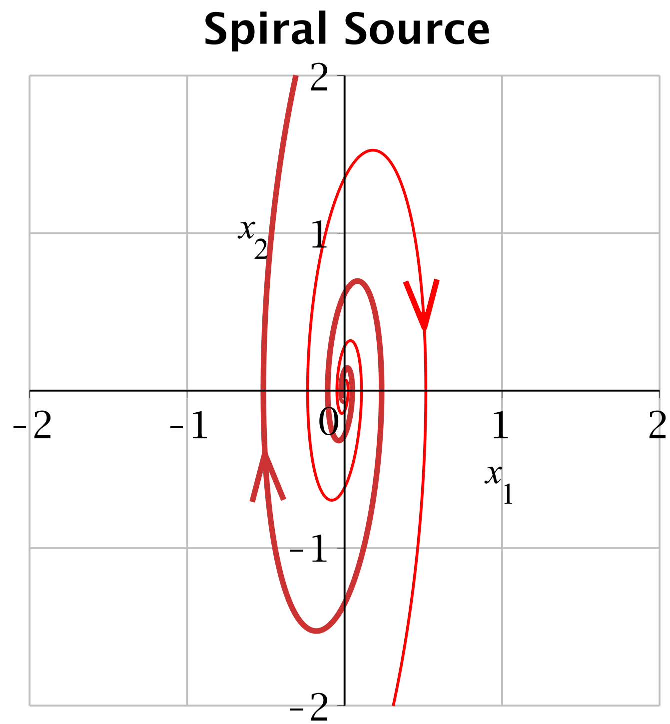

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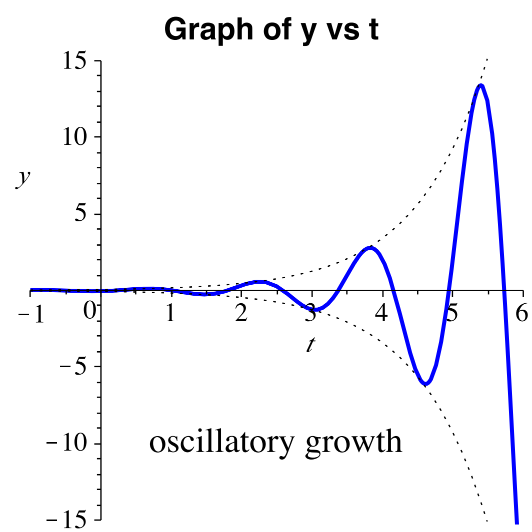

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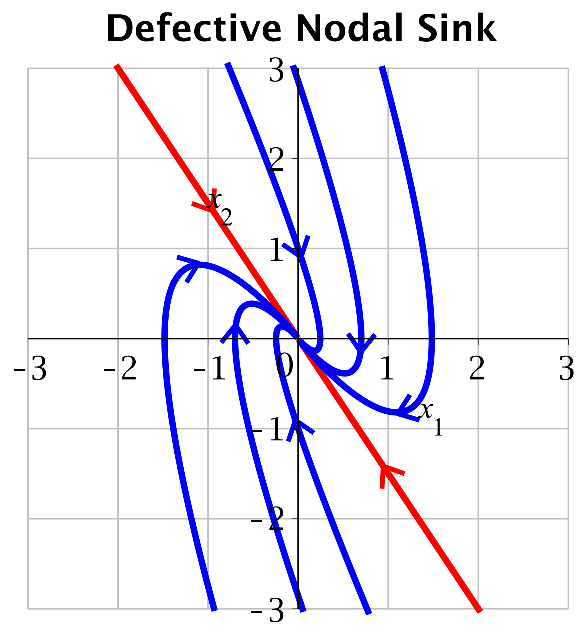

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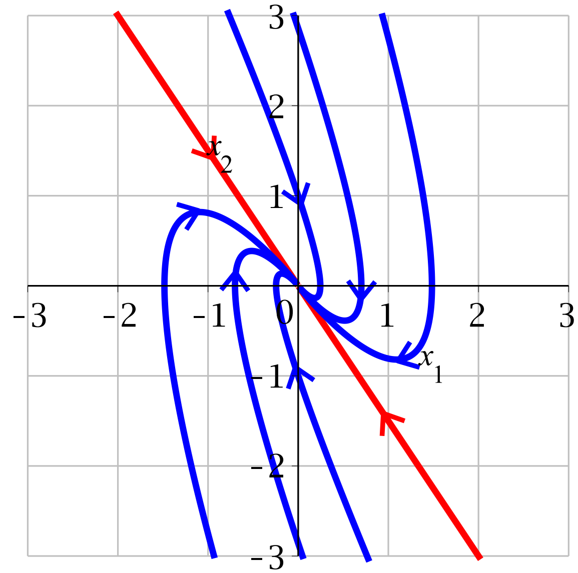

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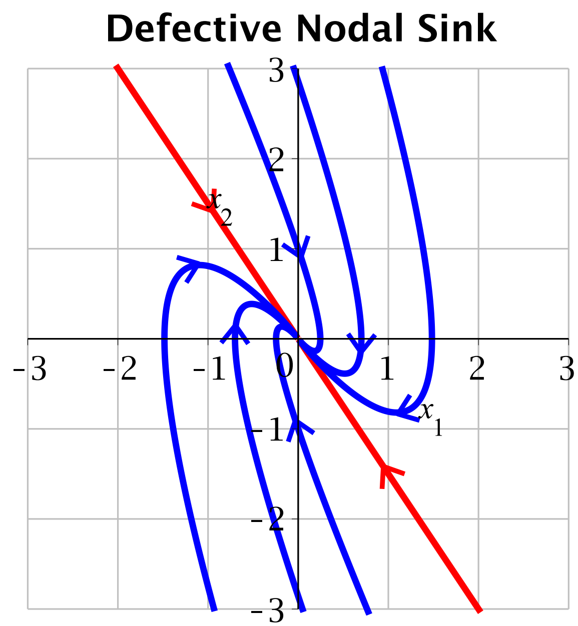

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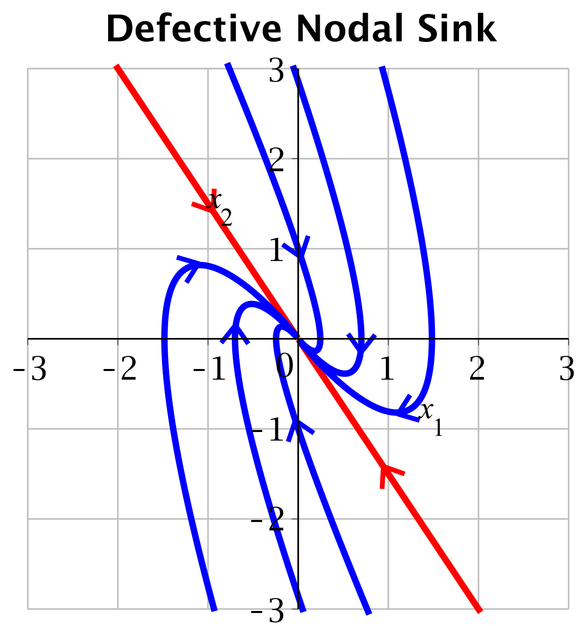

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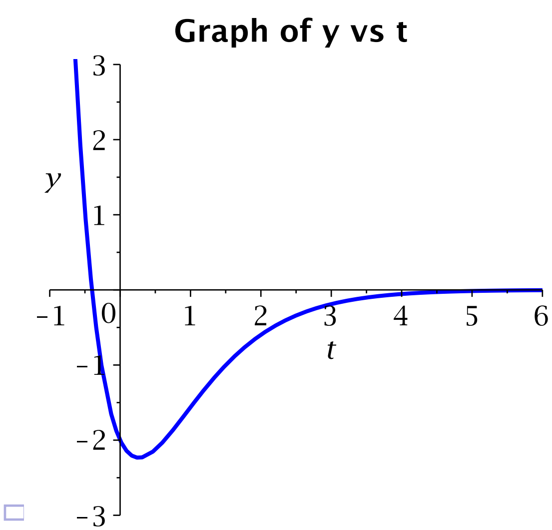

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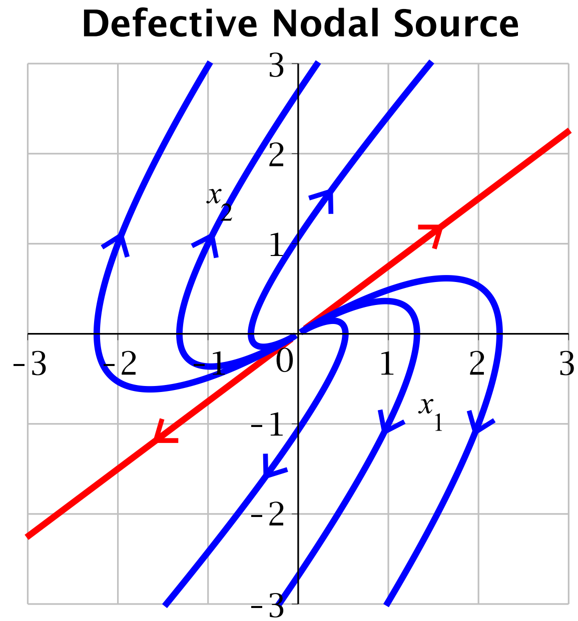

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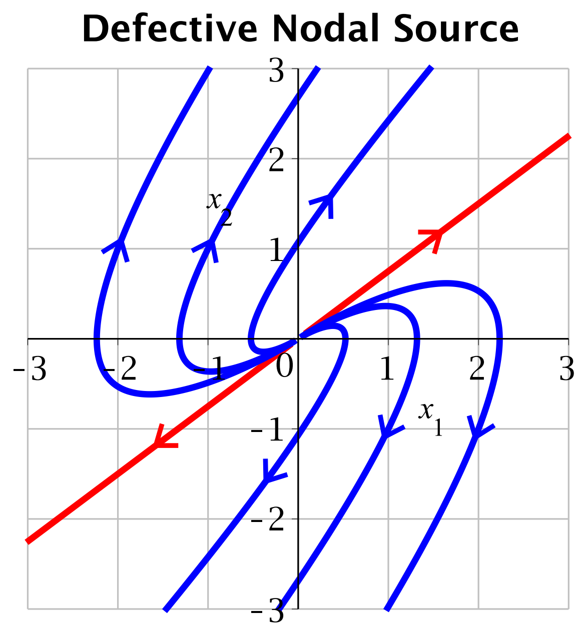

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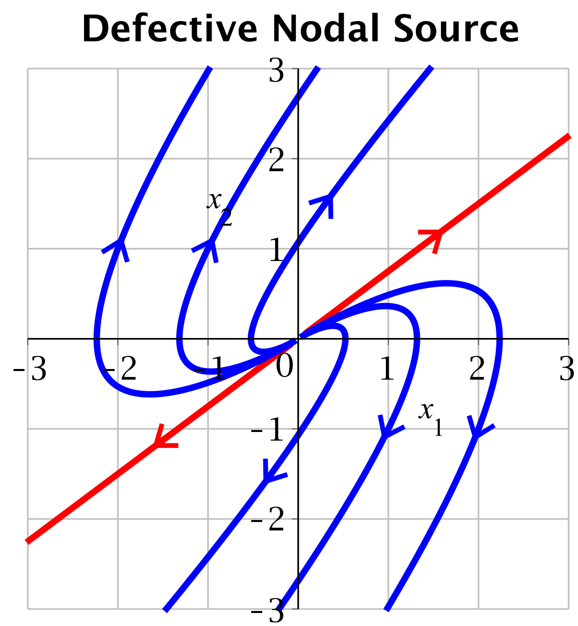

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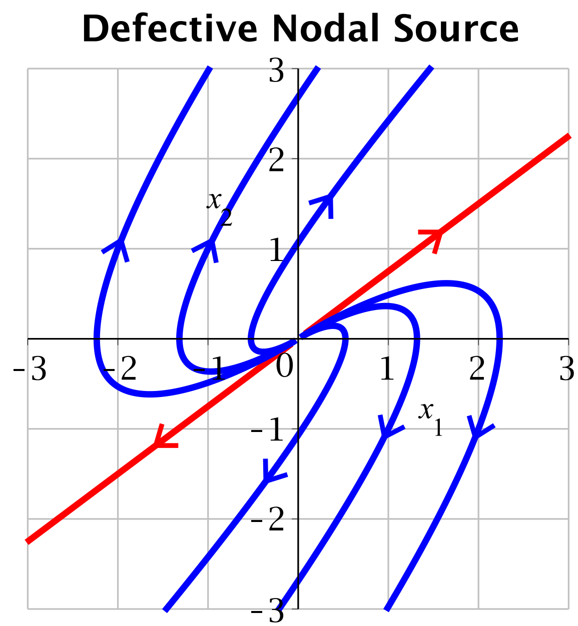

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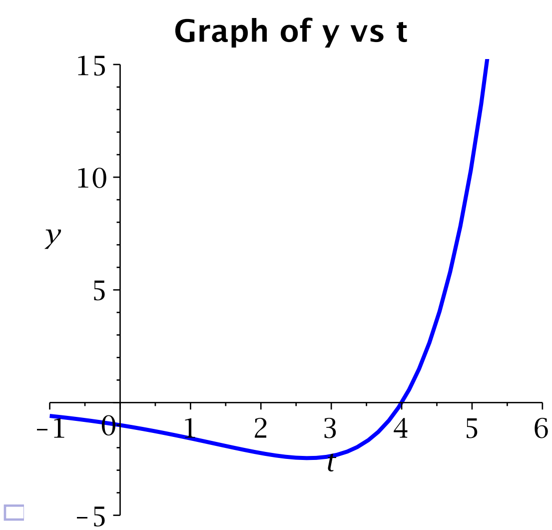

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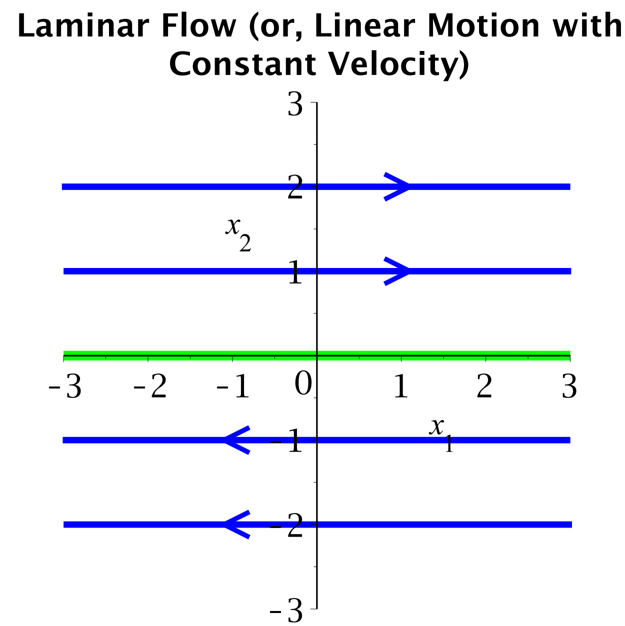

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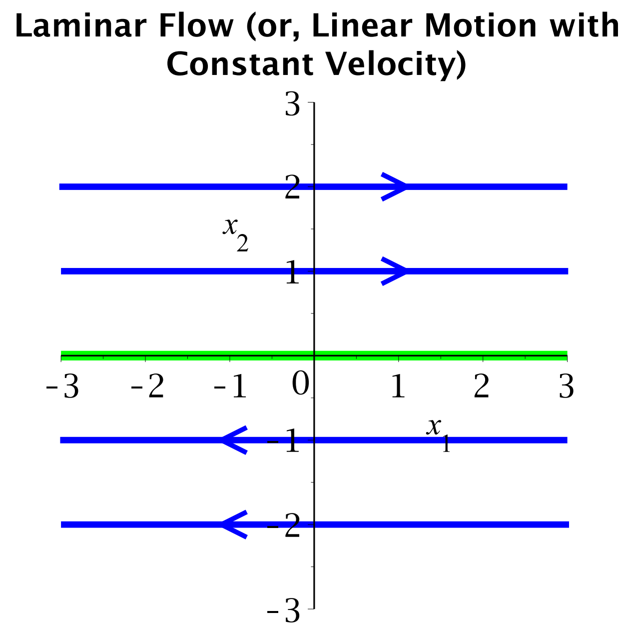

- Initial conditions $y(0) = 3, y'(0) = -\frac{1}{2}$

$$\begin{cases} C_1 = 3 \\ C_2 = -\frac{1}{2} \end{cases}$$

Example: $y'' = 0$

- Eigenvalues: $\lambda^2 = 0$

$$\lambda_1 = \lambda_2 = 0$$

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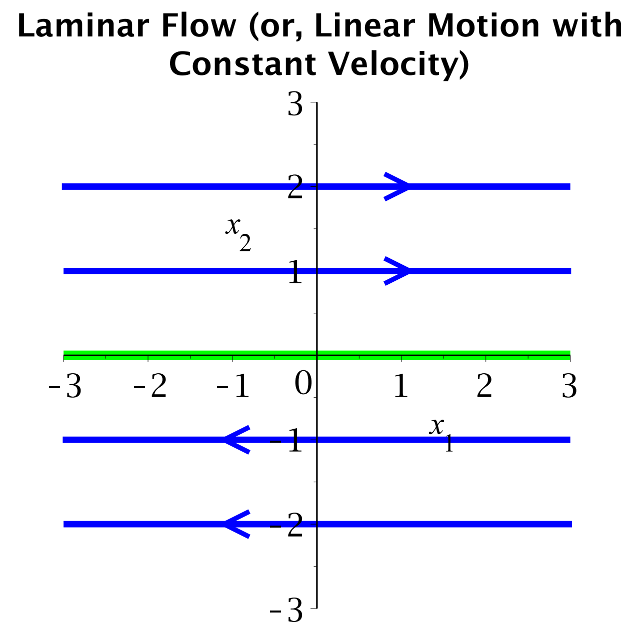

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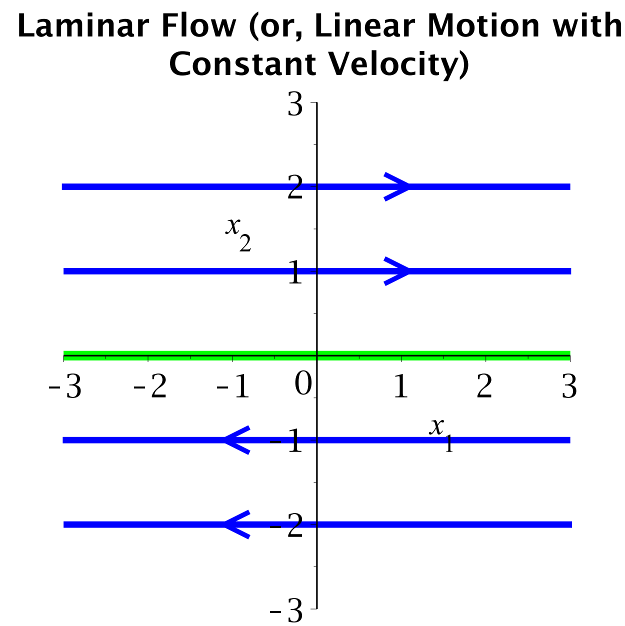

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