Nonhomogeneous Linear Systems of Differential Equations with Constant Coefficients

Objective: Solve

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

where A is an $n \times n$ constant coefficient matrix A and $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$ is a given vector function.

The unknown is $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

Solution Formula Using Fundamental Matrix: Suppose that $M(t)$ is a fundamental matrix solution of the corresponding homogeneous system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$; in other words,

- $M(t)$ satisfies $M'(t) = AM(t)$; that is, every column of $M(t)$ solves the homogeneous system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$;
- $M(t)$ is an invertible matrix for every t; that is, the n columns of $M(t)$ are linearly independent.

The general solutions of the nonhomogeneous system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$ are

$$\vec{\mathbf{x}}(t) = M(t) \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} + \int^t M(t)M(s)^{-1}\,\vec{\mathbf{f}}(s)ds,$$

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

is given by

$$\vec{\mathbf{x}}(t) = M(t)M(t_0)^{-1}\,\vec{\mathbf{x}}_0 + \int_{t_0}^t M(t)M(s)^{-1}\,\vec{\mathbf{f}}(s)ds.$$

Solution Formula Using Matrix Exponential: The general solutions of the nonhomogeneous system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$ are

$$\vec{\mathbf{x}}(t) = e^{tA} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} + \int^t e^{(t-s)A}\,\vec{\mathbf{f}}(s)ds,$$

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

is given by

$$\vec{\mathbf{x}}(t) = e^{(t-t_0)A}\,\vec{\mathbf{x}}_0 + \int_{t_0}^{t} e^{(t-s)A}\,\vec{\mathbf{f}}(s)ds.$$

Solution Method by Decoupling: If A is diagonalizable (i.e., $A = PDP^{-1}$ with an invertible P and a diagonal D), then the system can be decoupled by setting $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

$$\frac{d\vec{\mathbf{u}}}{dt} = D\vec{\mathbf{u}} + P^{-1}\vec{\mathbf{f}}(t).$$

EXAMPLE. Solve the initial value probelm

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 6/7 & -15/14 \\ -5/7 & 37/14 \end{bmatrix}\vec{\mathbf{x}} + \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Solution 1 (Use a fundamental matrix): First find eigenvalues and eigenvectors of A.

The eigenvalues of A are $\lambda_1 = 1/2, \lambda_2 = 3$.

Vector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 1/2$.

Vector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = 3$.

Thus,

$$M(t) = [\,e^{\lambda_1 t}\mathbf{v}_1 \quad e^{\lambda_2 t}\mathbf{v}_2\,] = \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}$$

is a fundamental matrix for the homogeneous system $\vec{\mathbf{x}}\,'(t) = A\vec{\mathbf{x}}(t)$.

The solution to the initial value problem is given by

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= M(t)M(0)^{-1}\vec{\mathbf{x}}(0) + \int_0^t M(t)M(s)^{-1}\,\vec{\mathbf{f}}(s)ds \\
&= \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\begin{bmatrix} 3e^{s/2} & -e^{3s} \\ e^{s/2} & 2e^{3s} \end{bmatrix}^{-1}\begin{bmatrix} e^{2s} \\ e^{-s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} 2e^{3s/2} + e^{-3s/2} \\ -e^{-s} + 3e^{-4s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} \frac{4}{3}(e^{3t/2} - 1) - \frac{2}{3}(e^{-3t/2} - 1) \\ (e^{-t} - 1) - \frac{3}{4}(e^{-4t} - 1) \end{bmatrix} \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} - \frac{2}{7}e^{t/2} + \frac{1}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} - \frac{2}{21}e^{t/2} - \frac{1}{14}e^{3t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}.
\end{aligned}$$

Solution 2 (Use decoupling): The geiven coefficient matrix A is diagonalizable: $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Set $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

$$\frac{d\vec{\mathbf{u}}}{dt} = D\vec{\mathbf{u}} + P^{-1}\vec{\mathbf{f}}(t) = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}\vec{\mathbf{u}} + \begin{bmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}, \quad \vec{\mathbf{u}}(0) = P^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

or, equivalently,

$$\begin{aligned} u_1' &= \frac{1}{2}u_1 + \frac{2}{7}e^{2t} + \frac{1}{7}e^{-t}, \quad u_1(0) = 1, \\ u_2' &= 3u_2 - \frac{1}{7}e^{2t} + \frac{3}{7}e^{-t}, \quad u_2(0) = -1. \end{aligned}$$

These two equations can be solved separately (the method of integrating factor and the method of undetermined coefficients both work in this case). The solutions are

$$\begin{aligned} u_1(t) &= \frac{4}{21}e^{2t} - \frac{2}{21}e^{-t} + \frac{19}{21}e^{t/2}, \\ u_2(t) &= \frac{1}{7}e^{2t} - \frac{3}{28}e^{-t} - \frac{29}{28}e^{3t}. \end{aligned}$$

Finally, the solution to the original problem is given by

$$\begin{aligned} \vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t) &= P\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{4}{21}e^{2t} - \frac{2}{21}e^{-t} + \frac{19}{21}e^{t/2} \\ \frac{1}{7}e^{2t} - \frac{3}{28}e^{-t} - \frac{29}{28}e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}. \end{aligned}$$

Solution 3 (Use matrix exponential): First find e^{tA} using a fundamental matrix: $e^{tA} = M(t)M(0)^{-1}$. Or, one can also find e^{tA} by diagonalization:

$$e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{bmatrix} = \begin{bmatrix} \frac{6}{7}e^{t/2} + \frac{1}{7}e^{3t} & \frac{3}{7}e^{t/2} - \frac{3}{7}e^{3t} \\ \frac{2}{7}e^{t/2} - \frac{2}{7}e^{3t} & \frac{1}{7}e^{t/2} + \frac{6}{7}e^{3t} \end{bmatrix}.$$

The solution to the initial value problem is given by

$$\vec{\mathbf{x}}(t) = e^{tA}\vec{\mathbf{x}}(0) + \int_0^t e^{(t-s)A}\vec{\mathbf{f}}(s)ds$$

$$
\begin{aligned}
&= \begin{bmatrix} \frac{6}{7}e^{t/2} + \frac{1}{7}e^{3t} & \frac{3}{7}e^{t/2} - \frac{3}{7}e^{3t} \\ \frac{2}{7}e^{t/2} - \frac{2}{7}e^{3t} & \frac{1}{7}e^{t/2} + \frac{6}{7}e^{3t} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
&\quad + \int_0^t \begin{bmatrix} \frac{6}{7}e^{(t-s)/2} + \frac{1}{7}e^{3(t-s)} & \frac{3}{7}e^{(t-s)/2} - \frac{3}{7}e^{3(t-s)} \\ \frac{2}{7}e^{(t-s)/2} - \frac{2}{7}e^{3(t-s)} & \frac{1}{7}e^{(t-s)/2} + \frac{6}{7}e^{3(t-s)} \end{bmatrix} \begin{bmatrix} e^{2s} \\ e^{-s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{6}{7}e^{t/2+3s/2} + \frac{1}{7}e^{3t-s} + \frac{3}{7}e^{t/2-3s/2} - \frac{3}{7}e^{3t-4s} \\ \frac{2}{7}e^{t/2+3s/2} - \frac{2}{7}e^{3t-s} + \frac{1}{7}e^{t/2-3s/2} + \frac{6}{7}e^{3t-4s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} \\
&\quad + \begin{bmatrix} \frac{6}{7}e^{t/2} \cdot \frac{2}{3}(e^{3t/2} - 1) + \frac{1}{7}e^{3t}(1 - e^{-t}) + \frac{3}{7}e^{t/2} \cdot \frac{2}{3}(1 - e^{-3t/2}) - \frac{3}{7}e^{3t} \cdot \frac{1}{4}(1 - e^{-4t}) \\ \frac{2}{7}e^{t/2} \cdot \frac{2}{3}(e^{3t/2} - 1) - \frac{2}{7}e^{3t}(1 - e^{-t}) + \frac{1}{7}e^{t/2} \cdot \frac{2}{3}(1 - e^{-3t/2}) + \frac{6}{7}e^{3t} \cdot \frac{1}{4}(1 - e^{-4t}) \end{bmatrix} \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} - \frac{2}{7}e^{t/2} + \frac{1}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} - \frac{2}{21}e^{t/2} - \frac{1}{14}e^{3t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}.
\end{aligned}
$$

EXERCISES

[1] Solve

$$\begin{aligned} x_1' &= 5x_1 - 3x_2 + 8, \\ x_2' &= x_1 + x_2 + 32t, \\ x_1(0) &= 2, x_2(0) = 0. \end{aligned}$$

[2] Solve

$$\begin{aligned} x_1' &= -7x_1 - 9x_2 + 9x_3 + e^{-t}, \\ x_2' &= 3x_1 + 5x_2 - 3x_3 + 2e^{-t} + e^{t}, \\ x_3' &= -3x_1 - 3x_2 + 5x_3 + 3e^{t}, \\ x_1(0) &= 1, x_2(0) = 0, x_3(0) = 0. \end{aligned}$$

[3] Solve

$$\begin{aligned} x_1' &= -5x_1 - 8x_2 + 4x_3, \\ x_2' &= 2x_1 + 3x_2 - 2x_3 + e^{-t}, \\ x_3' &= 6x_1 + 14x_2 - 5x_3 + 9t, \\ x_1(0) &= 2, x_2(0) = 1, x_3(0) = 1. \end{aligned}$$

Answers:

[1] $x_1(t) = -10 - 12t + 3e^{4t} + 9e^{2t}$,
$x_2(t) = -10 - 20t + e^{4t} + 9e^{2t}$

[2] $x_1(t) = -9e^{t} + \frac{4}{3}e^{2t} + \frac{26}{3}e^{-t} + 9te^{-t}$,
$x_2(t) = 2e^{t} + \frac{5}{3}e^{2t} - \frac{11}{3}e^{-t} - 3te^{-t}$,
$x_3(t) = -6e^{t} + 3e^{2t} + 3e^{-t} + 3te^{-t}$

[3] $x_1(t) = -\frac{8}{3} + 4t + (4 + 6t)e^{-t} + (\frac{2}{3} - 18t)e^{-3t}$,
$x_2(t) = \frac{4}{3} - 2t - 2te^{-t} + (-\frac{1}{3} + 9t)e^{-3t}$,
$x_3(t) = \frac{1}{3} + t + (\frac{11}{2} + 2t)e^{-t} + (-\frac{29}{6} + 9t)e^{-3t}$