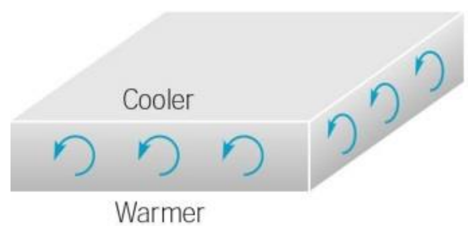
Fluid Dynamics

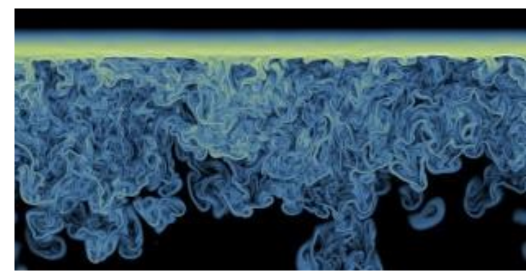
Asymptotically
stable
trivial equilibrium.
Fluid at Rest.

Steady
convective rolls
(clockwise &
counterclockwise)

Complicated
turbulent flows.
Impossibility of
forecasting.

ΔT





Lorenz Equations

r=1

r=470/19

r

Asymptotically
stable
equilibrium (0,0,0)

(0,0,0) becomes
unstable.
Two new asympt
stable equilibria.

All equilibria are
unstable.
Chaotic solutions.
Unpredictable behavior.
Strange Attractor.

Lorenz Equations

$$\begin{cases} \dfrac{dx}{dt} = \sigma(-x+y) \\ \dfrac{dy}{dt} = r x - y - xz \\ \dfrac{dz}{dt} = -b z + xy \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = 10(-x+y) \\ \dfrac{dy}{dt} = r x - y - xz \\ \dfrac{dz}{dt} = -\dfrac{8}{3} z + xy \end{cases}$$

<u>Summary</u> Let $\sigma = 10$, $b = \frac{8}{3}$.

Increasing r, the asymptotics of the system will change from an attractive equilibrium to a strange attractor with chaotic dynamics.

· Find equilibria

eg. $r=5$

$$\begin{cases} \dfrac{dx}{dt} = 10(-x+y) \\ \dfrac{dy}{dt} = 5x - y - xz \\ \dfrac{dz}{dt} = -\dfrac{8}{3}z + xy \end{cases}$$

$$\begin{cases} ① & 10(-x+y)=0 \\ ② & 5x-y-xz=0 \\ ③ & -\dfrac{8}{3}z+xy=0 \end{cases}$$

① $\Rightarrow$ ④ $y=x$

③ $\Rightarrow$ ⑤ $z=\frac{3}{8}xy=\frac{3}{8}x^2$

Plug ④⑤ in ②:

$5x-x-x(\frac{3}{8}x^2)=0$, $x(4-\frac{3}{8}x^2)=0$

$x=0$, $x=\sqrt{32/3}=\frac{4}{3}\sqrt{6}$, $x=-\sqrt{32/3}=-\frac{4}{3}\sqrt{6}$

- $x=0 \overset{④⑤}{\Rightarrow} y=0,\ z=0$.
- $x=\frac{4}{3}\sqrt{6} \Rightarrow y=\frac{4}{3}\sqrt{6},\ z=\frac{3}{8}\left(\frac{4}{3}\sqrt{6}\right)^2=4$.
- $x=-\frac{4}{3}\sqrt{6} \Rightarrow y=-\frac{4}{3}\sqrt{6},\ z=\frac{3}{8}\left(-\frac{4}{3}\sqrt{6}\right)^2=4$.
- Three Equilibria: $(0,0,0)$, $(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}, 4)$, $(-\frac{4}{3}\sqrt{6}, -\frac{4}{3}\sqrt{6}, 4)$.

Construct Linear Approximating System Near an Equilibrium.

$$\begin{cases} \dfrac{dx}{dt} = 10(-x+y) \\ \dfrac{dy}{dt} = r\,x-y-xz \\ \dfrac{dz}{dt} = -\dfrac{8}{3}z+xy \end{cases}$$

$$f_1(x,y,z) = 10(-x+y)$$
$$f_2(x,y,z) = r\,x-y-xz$$
$$f_3(x,y,z) = -\frac{8}{3}z+xy$$

- Let (a,b,c) be an equilibrium.
- The Linear Approximating System Near Equilibrium (a,b,c):

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} J \text{ evaluated at } (a,b,c) \end{bmatrix}\begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}.$$

- Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ r-z & -1 & -x \\ y & x & -\frac{8}{3} \end{bmatrix}$$

When $r<1$, the system has a unique equilibrium $(0,0,0)$, which is asympt. stable

e.g.

$r=\frac{1}{2}$

$$\begin{cases} \frac{dx}{dt} = 10(-x+y) \\ \frac{dy}{dt} = \frac{1}{2}x - y - xz \\ \frac{dz}{dt} = -\frac{8}{3}z + xy \end{cases}$$

Unique Equilibrium: $(x,y,z)=(0,0,0)$

Linear Approximating System near $(0,0,0)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

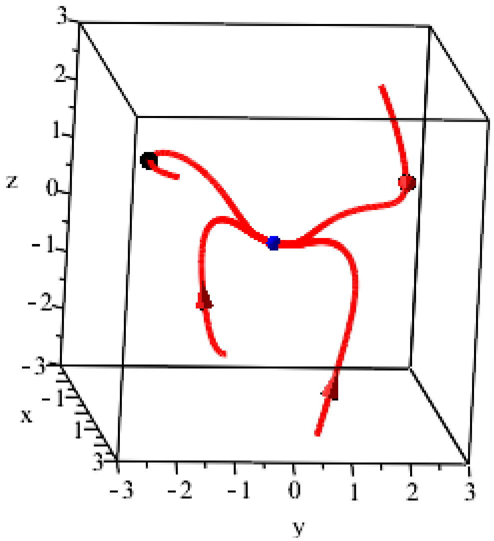
Eigenvalues:

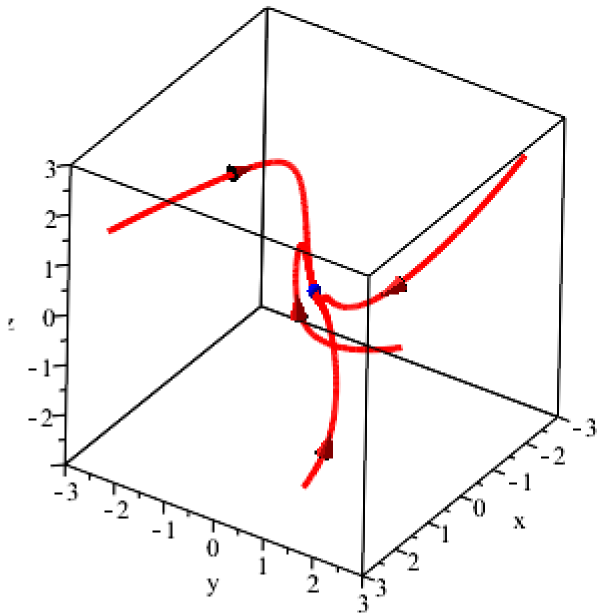
$$\lambda_1 = -\frac{8}{3} = -2.666\cdots\cdots$$

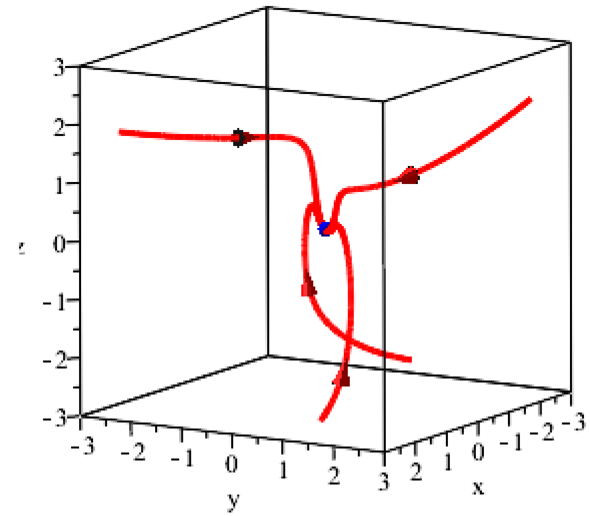
$$\lambda_2 = \frac{-11+\sqrt{101}}{2} = -0.475062190\cdots$$

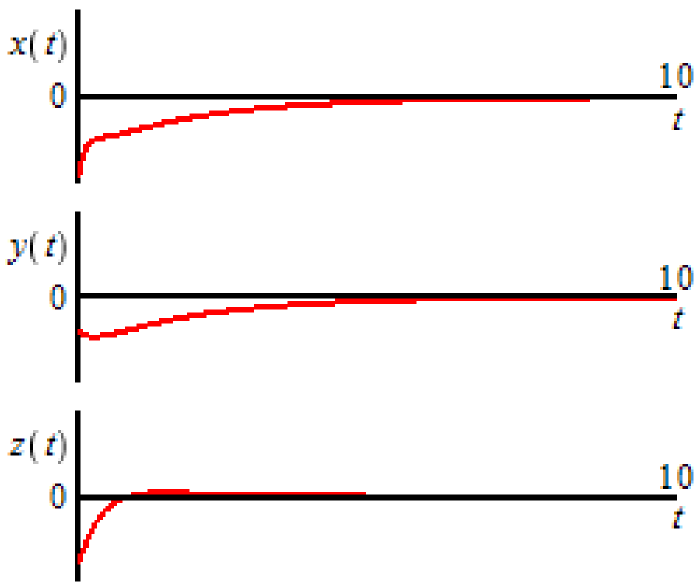
$$\lambda_3 = \frac{-11-\sqrt{101}}{2} = -10.52493781\cdots\cdots$$

$r < 1: \quad r = \frac{1}{2}$









When $r=1$, the equilibrium $(0,0,0)$ becomes degenerate.

$r=1$

$$\begin{cases} \dfrac{dx}{dt} = 10(-x+y) \\ \dfrac{dy}{dt} = 1\cdot x - y - xz \\ \dfrac{dz}{dt} = -\dfrac{8}{3}z + xy \end{cases}$$

Unique Equilibrium: $(x,y,z)=(0,0,0)$

Linear Approximating System near $(0,0,0)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = -\dfrac{8}{3}$

$\lambda_2 = 0$ (neutral eigenvalue)

$\lambda_3 = -11$

When $1 < r < \frac{470}{19} = 24.73684211\cdots$, there are three equilibria.

(0,0,0) becomes unstable, and

the two new equilibria are asymptotically stable.

e.g. $r=5$
$$\begin{cases} \frac{dx}{dt} = 10(-x+y) \\ \frac{dy}{dt} = 5x - y - xz \\ \frac{dz}{dt} = -\frac{8}{3}z + xy \end{cases}$$

Three Equilibria:

(0,0,0),

$\left(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}, 4\right), \left(-\frac{4}{3}\sqrt{6}, -\frac{4}{3}\sqrt{6}, 4\right)$.

- Linear Approximating System near (0,0,0):

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 5 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = -\frac{8}{3} = -2.666\cdots\cdots$

$\lambda_2 = \frac{1}{2}(-11+\sqrt{281}) = 2.8815\cdots\cdots$ positive

$\lambda_3 = \frac{1}{2}(-11-\sqrt{281}) = -13.8815\cdots\cdots$

- (0,0,0) is unstable.

When $1 < r < \frac{470}{19} = 24.73684211\cdots$, there are three equilibria.
$(0,0,0)$ becomes unstable, and
the two new equilibria are asymptotically stable.

e.g. $r=5$

$$\begin{cases} \frac{dx}{dt} = 10(-x+y) \\ \frac{dy}{dt} = 5x - y - xz \\ \frac{dz}{dt} = -\frac{8}{3}z + xy \end{cases}$$

Three Equilibria:
$(0,0,0)$,
$\left(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}, 4\right)$, $\left(-\frac{4}{3}\sqrt{6}, -\frac{4}{3}\sqrt{6}, 4\right)$.

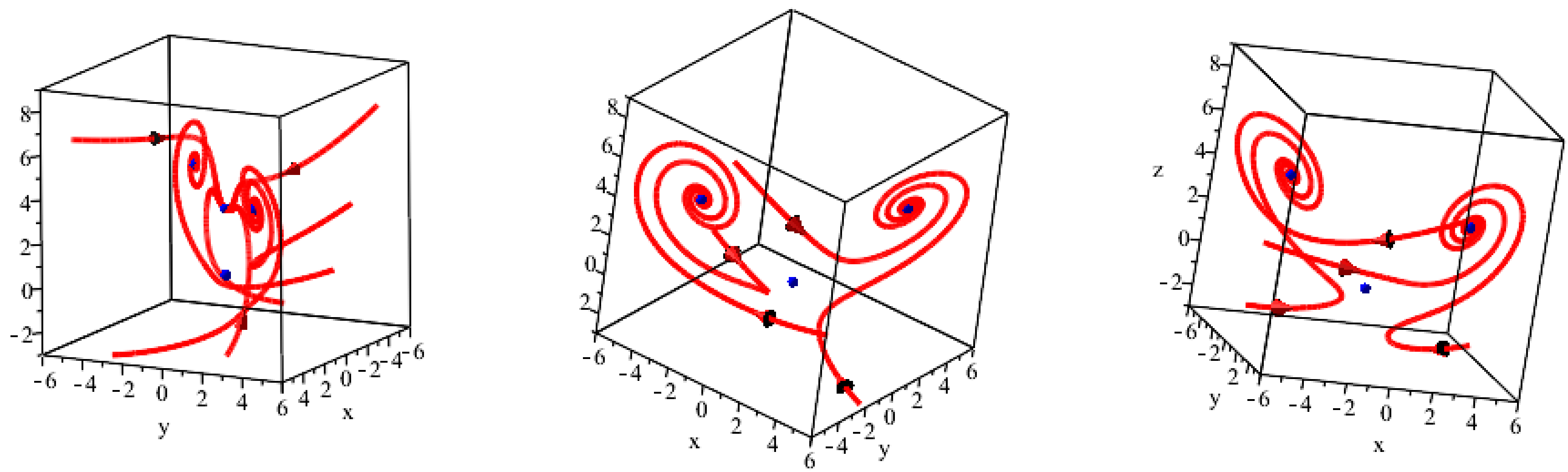
- Linear Approximating System near $\left(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}, 4\right)$:

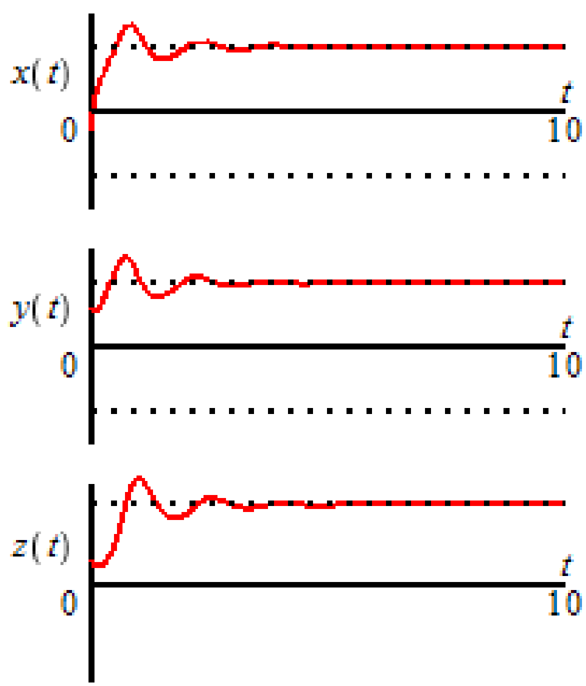
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\frac{4}{3}\sqrt{6} \\ \frac{4}{3}\sqrt{6} & \frac{4}{3}\sqrt{6} & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x - \frac{4}{3}\sqrt{6} \\ y - \frac{4}{3}\sqrt{6} \\ z - 4 \end{bmatrix}$$

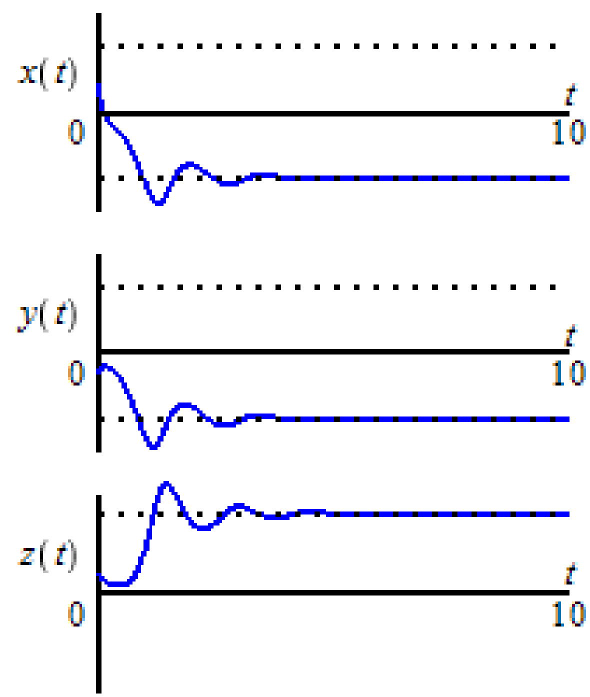
Eigenvalues:
$\lambda_1 = -11.809\cdots$
$\lambda_2 = -0.9287\cdots + 4.14758\cdots i$
$\lambda_3 = -0.9287\cdots - 4.14758\cdots i$

- $\left(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}, 4\right)$ is asymptotically stable.

$$1 < r < \frac{470}{19} = 24.73684211: \quad r = 5$$







When $r=\frac{470}{19}=24.73684211\cdots$, there are three equilibria.
$(0,0,0)$ is still unstable, and
the two nonzero equilibria become degenerate.

$r=\frac{470}{19}$

$$\begin{cases} \frac{dx}{dt}=10(-x+y) \\ \frac{dy}{dt}=\frac{470}{19}x-y-xz \\ \frac{dz}{dt}=-\frac{8}{3}z+xy \end{cases}$$

Three Equilibria:

$(0,0,0)$,

$\left(\frac{2}{57}\sqrt{51414}, \frac{2}{57}\sqrt{51414}, \frac{451}{19}\right)$,

$\left(-\frac{2}{57}\sqrt{51414}, -\frac{2}{57}\sqrt{51414}, \frac{451}{19}\right)$.

• Lin. Approx. System Near $(0,0,0)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -10 & 10 & 0 \\ \frac{470}{19} & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues: $\lambda_1=-8/3=-2.666\cdots$

$\lambda_2=-\frac{11}{2}+\frac{43}{38}\sqrt{209}=10.859\cdots>0$

$\lambda_3=-\frac{11}{2}-\frac{43}{38}\sqrt{209}=-21.859\cdots$

• Lin. Approx. System Near $\left(\frac{2}{57}\sqrt{51414}, \frac{2}{57}\sqrt{51414}, \frac{451}{19}\right)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\frac{2}{57}\sqrt{51414} \\ \frac{2}{57}\sqrt{51414} & \frac{2}{57}\sqrt{51414} & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x-\frac{2}{57}\sqrt{51414} \\ y-\frac{2}{57}\sqrt{51414} \\ z-\frac{451}{19} \end{bmatrix}$$

Eigenvalues:

$\lambda_1=-\frac{41}{3}$

$\lambda_2=\frac{4}{19}\sqrt{2090}\,i$

$\lambda_3=-\frac{4}{19}\sqrt{2090}\,i$

$\lambda_{2,3}$ are neutral.

When $r > \frac{470}{19} = 24.73684211\cdots$, there are three equilibria.

- $(0,0,0)$ is still unstable, and

the other two equilibria also become unstable.

- A Strange attractor appears. The system exhibits chaotic dynamics

e.g. $r=28$

$$\begin{cases} \frac{dx}{dt} = 10(-x+y) \\ \frac{dy}{dt} = 28x - y - xz \\ \frac{dz}{dt} = -\frac{8}{3}z + xy \end{cases}$$

Three Equilibria:

$(0,0,0)$,

$(6\sqrt{2}, 6\sqrt{2}, 27)$, $(-6\sqrt{2}, -6\sqrt{2}, 27)$

- Linear Approximating System near $(0,0,0)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = -\frac{8}{3} = -2.666\cdots\cdots$

$\lambda_2 = \frac{1}{2}(-11+\sqrt{1201}) = 11.8277\cdots > 0$

$\lambda_3 = \frac{1}{2}(-11-\sqrt{1201}) = -22.8277\cdots$

- $(0,0,0)$ is unstable.

When $r > \frac{470}{19} = 24.73684211\cdots$, there are three equilibria.

- $(0,0,0)$ is still unstable, and the other two equilibria also become unstable.
- A **Strange attractor** appears. The system exhibits **chaotic dynamics**.

e.g. $r=28$

$$\begin{cases} \frac{dx}{dt} = 10(-x+y) \\ \frac{dy}{dt} = 28x - y - xz \\ \frac{dz}{dt} = -\frac{8}{3}z + xy \end{cases}$$

Three Equilibria:

$(0,0,0)$,

$(6\sqrt{2}, 6\sqrt{2}, 27)$, $(-6\sqrt{2}, -6\sqrt{2}, 27)$

- Linear Approximating System near $(6\sqrt{2}, 6\sqrt{2}, 27)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3} \end{bmatrix}\begin{bmatrix} x-6\sqrt{2} \\ y-6\sqrt{2} \\ z-27 \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = -13.85$

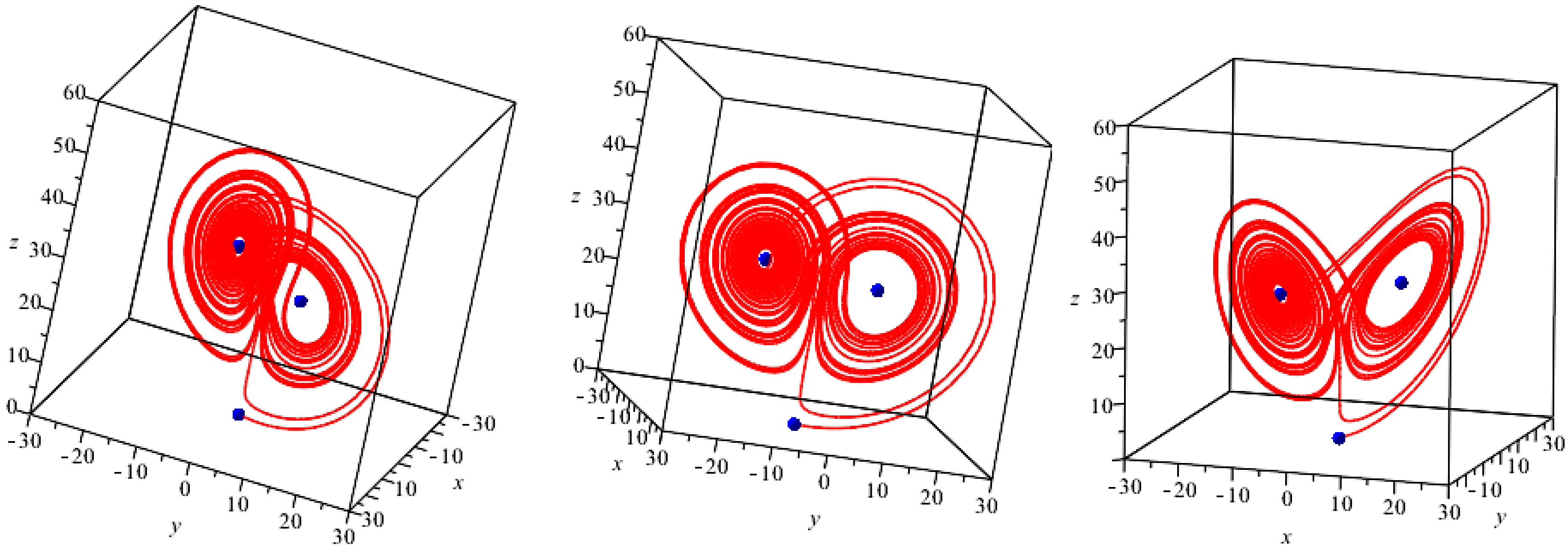
$\lambda_2 = 0.09\cdots + 10.19\cdots i$

$\lambda_3 = 0.09\cdots - 10.19\cdots i$

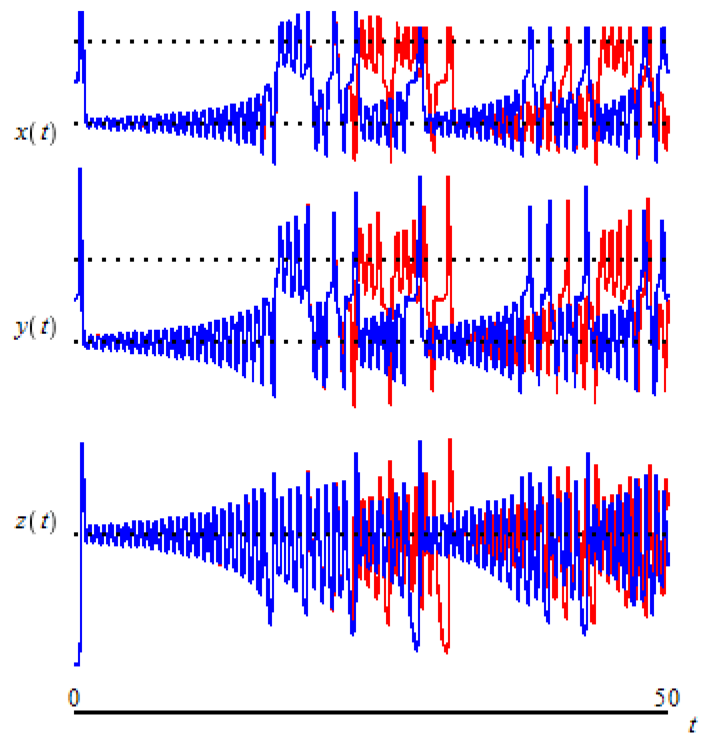
- $(6\sqrt{2}, 6\sqrt{2}, 27)$ is unstable.

$$r > \frac{470}{19} = 24.73684211: \quad r = 28$$

Strange Attractor



Chaos: Unpredictable solutions.
Sensitive dependence of solutions on the initial conditions.



Red solution:

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1$$

Blue solution:

$$x(0) = 0.1, y(0) = 0.101, z(0) = 0.1$$