Local Phase Portrait of Nonlinear Systems Near Equilibria

[1] Consider

$$\begin{cases} x_1' = 60x_1 - 4x_1^2 - 3x_1x_2, \\ \\ x_2' = 42x_2 - 3x_1x_2 - 2x_2^2. \end{cases} \tag{$*$}$$

(a) Find all equilibrium solutions of the system $(*)$.

(b) For each equilibrium point,

- give the linear approximating system near the equilibrium;
- sketch the phase portrait of the linear approximating system;
- sketch the local phase portrait of the original nonlinear system $(*)$ near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(*)$.

[2] Consider

$$\begin{cases} x_1' = 5x_1 - x_1^2 - x_1x_2, \\ \\ x_2' = -2x_2 + x_1x_2. \end{cases} \tag{$**$}$$

(a) Find all equilibrium solutions of the system $(**)$.

(b) For each equilibrium point,

- give the linear approximating system near the equilibrium;
- sketch the phase portrait of the linear approximating system;
- sketch the local phase portrait of the original nonlinear system $(**)$ near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(**)$.

[3] Consider

$$\begin{cases} x_1' = -2x_1 - 2x_2, \\ \\ x_2' = -x_1 - x_2 - x_1^4. \end{cases} \tag{$**$}$$

(a) Find all equilibrium solutions of the system $(**)$.

(b) For each equilibrium point,

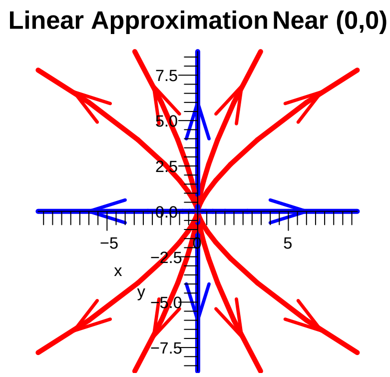
- give the linear approximating system near the equilibrium;
- sketch the phase portrait of the linear approximating system;
- sketch the local phase portrait of the original nonlinear system $(**)$ near the equilibrium;
- determine whether the equilibrium is stable or unstable with respect to the nonlinear system $(**)$.

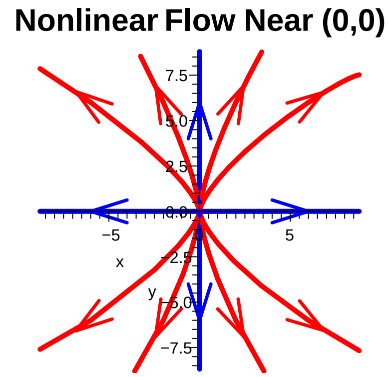
Turn over for the answers

Answers:

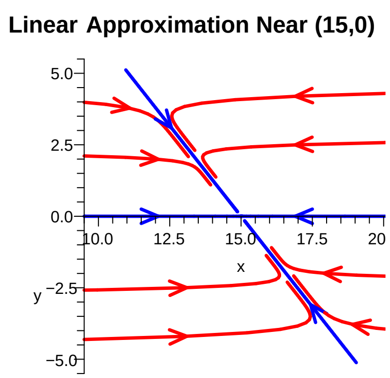
[1] (a) $(0,0), (15,0), (0,21), (6,12)$.

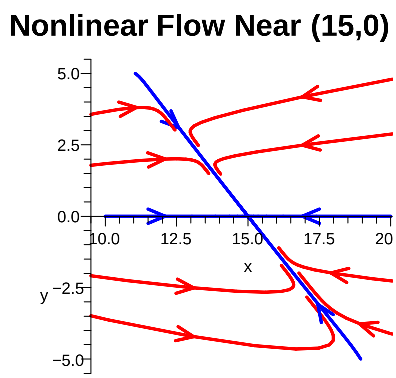
(b) Near $(0,0)$: the linear approximating system is $x_1' = 60x_1, x_2' = 42x_2$.
$(0,0)$ is a repulsive node.



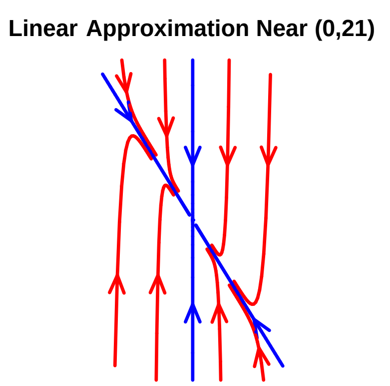


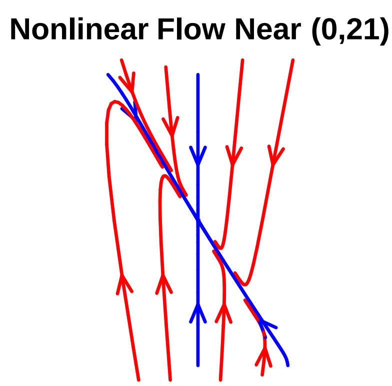
Near $(15,0)$: the linear approximating system is $x_1' = -60(x_1-15)-45x_2, x_2' = -3x_2$.
$(15,0)$ is an attractive node.





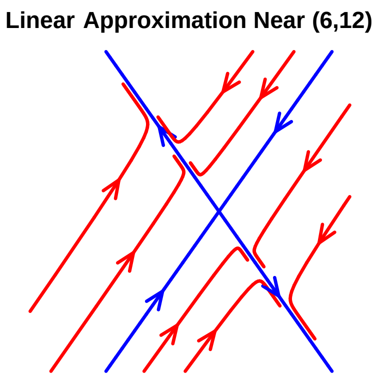
Near $(0,21)$: the linear approximating system is $x_1' = -3x_1, x_2' = -63x_1-42(x_2-21)$.
$(0,21)$ is an attractive node.

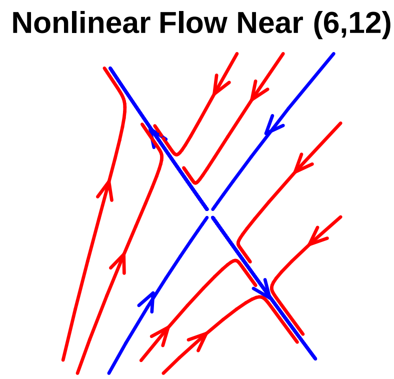




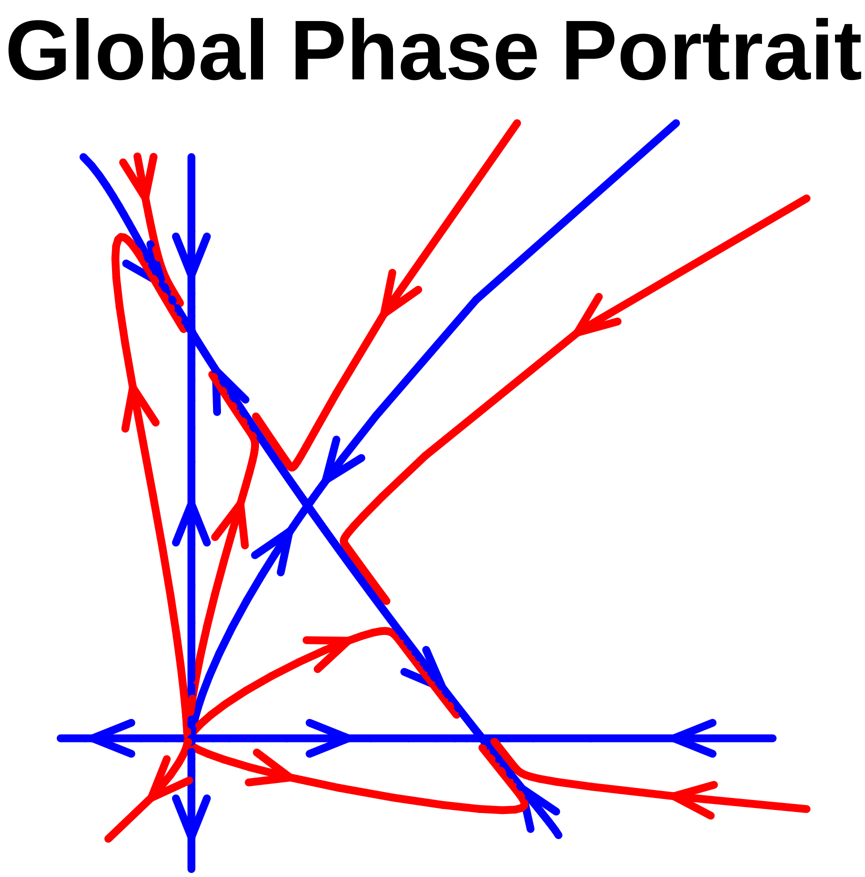
Near $(6, 12)$: the linear approximating system is $x_1' = -24(x_1 - 6) - 18(x_2 - 12)$, $x_2' = -36(x_1 - 6) - 24(x_2 - 12)$.

$(6, 12)$ is a saddle.





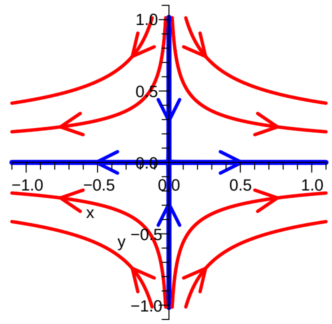
For your reference, the global phase portrait of the nonlinear system is included below:



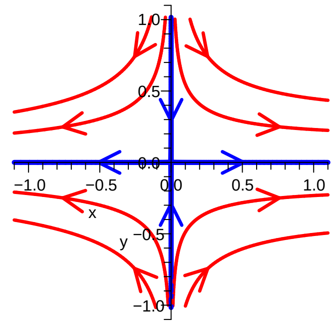
[2] (a) $(0,0),(5,0),(2,3)$.

(b) Near $(0,0)$: the linear approximating system is $x_1' = 5x_1, x_2' = -2x_2$.
$(0,0)$ is a saddle.

Linear Approximation Near (0,0)

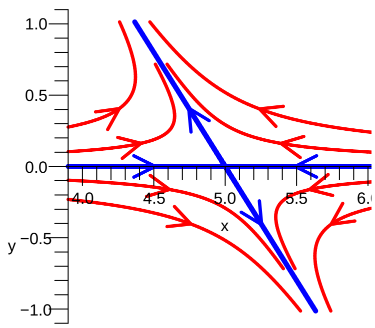


Nonlinear Flow Near (0,0)

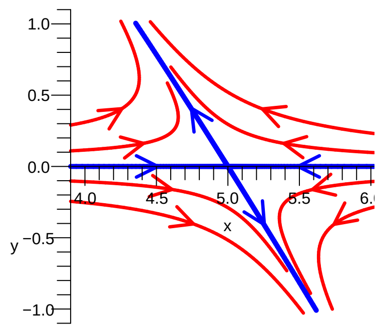


Near $(5,0)$: the linear approximating system is $x_1' = -5(x_1-5) - 5x_2, x_2' = 3x_2$.
$(5,0)$ is a saddle.

Linear Approximation Near (5,0)

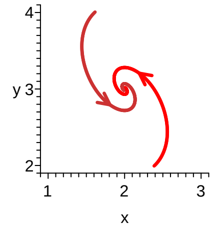


Nonlinear Flow Near (5,0)

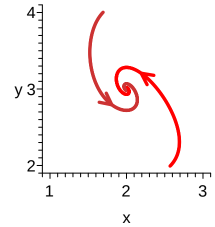


Near $(2,3)$: the linear approximating system is $x_1' = -2(x_1-2) - 2(x_2-3), x_2' = 3(x_1-2)$.
$(2,3)$ is an attractive spiral focus.

Linear Approximation Near (2,3)

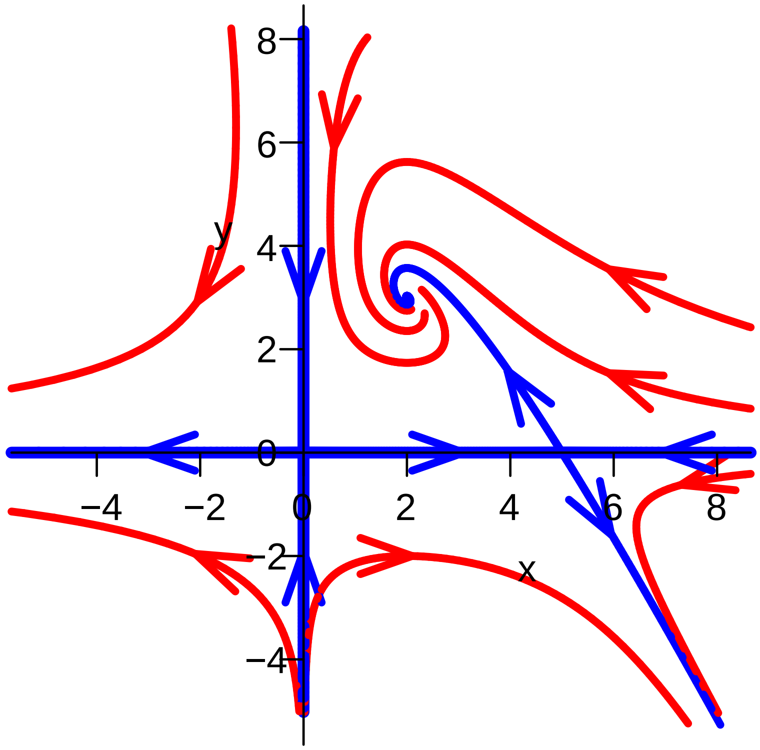


Nonlinear Flow Near (2,3)



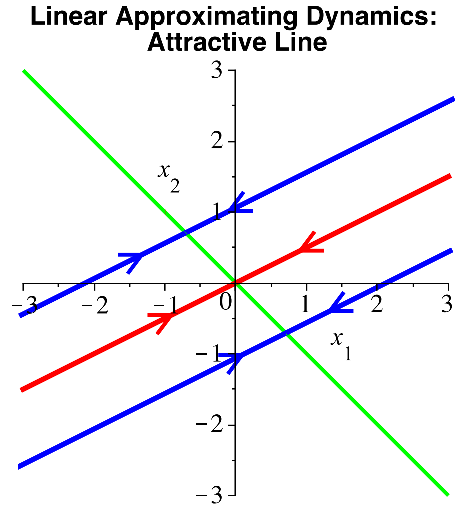
For your reference, the global phase portrait of the nonlinear system is included below:

Global Phase Portrait



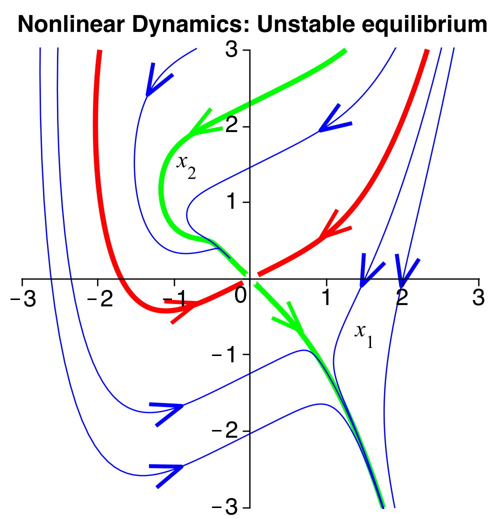
[3] (a) $(0, 0)$.

(b) Near $(0, 0)$:



the linear approximating system is $x_1' = -2x_1 - 2x_2, x_2' = -x_1 - x_2$.
The Jacobian matrix $J(0,0) = \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$.
This allows us to sketch the phase portrait for the linear approximating system, which is shown on the right.
In particular, $(0, 0)$ is a stable (but not asymptotically stable) equilibrium, with repsect to the linear approximating system.

This, however, does not automatically give the picture for the original nonlinear system. Because one of the eigenvalues of $J(0, 0)$ is $\lambda_1 = 0$, the local dynamics of the nonliear system near $(0, 0)$ may be significantly different from that of the linear system. The linear approximation alone is not enough to determine the nonlinear dynamics. In order to determine the local dynamics of the nonlinear system in such cases, we need to use more advanced techniques (the theory of center manifolds and so on). Indeed, it can be proved that $(0, 0)$ is an unstable equilibrium with respect to the original nonlinear system; hence, for this particular system, the local nonlinear dynamics is actually different from the linear dynamics. But we only find this after a much more complicated nonlinear analysis, which we did not (and will not) cover in this course.



In summary, in the present exercise, linear analysis was helpful in giving a partial information, but unfortunately was not enough to answer all the questions asked. Well, I still hope that you learned a thing or two by attempting to solve this exercise.

For your reference and viewing pleasure, the phase portrait of the nonlinear system of the present exercise is shown on the right.