Convolution $(f * g)(t)$

Definition Given $f(t)$ & $g(t)$ on $0 \le t < \infty$.

the **convolution** of f & g is :

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)\, g(\tau)\, d\tau .$$

Theorem If $f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$, $g(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} G(s)$

then

$$(f * g)(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)\, G(s) .$$

<u>Example</u> Let $f(t) = e^{t}$, $g(t) = e^{3t}$.

(a) Find $(f*g)(t)$, using the definition of convolution.

(b) Find $(g*f)(t)$, - - - - - - - - -

(c) Find $(1*f)(t)$. - - - - - - - - -

(d) Verify that $\mathcal{L}\{f*g(t)\} = F(s)\,G(s)$.

<u>Solution</u>

(a) $$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t e^{t-\tau}\cdot e^{3\tau}\,d\tau = \int_0^t e^{t}\cdot e^{2\tau}\,d\tau$$

$$= \left[\frac{1}{2}e^{t}e^{2\tau}\right]_{\tau=0}^{\tau=t} = \frac{1}{2}e^{3t} - \frac{1}{2}e^{t}.$$

(b) $$(g*f)(t) = \int_0^t g(t-\tau)\,f(\tau)\,d\tau = \int_0^t e^{3(t-\tau)}e^{\tau}\,d\tau = \int_0^t e^{3t}\cdot e^{-2\tau}\,d\tau$$

$$= \left[-\frac{1}{2}e^{3t}\cdot e^{-2\tau}\right]_{\tau=0}^{\tau=t} = -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t}.$$

(c) $$(1*f)(t) = \int_0^t 1\cdot f(\tau)\,d\tau = \int_0^t e^{\tau}\,d\tau = e^{t}-1.$$

(d) $$\mathcal{L}\{f*g(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{3t} - \frac{1}{2}e^{t}\right\} = \frac{1}{2}\frac{1}{s-3} - \frac{1}{2}\cdot\frac{1}{s-1} = \frac{1}{2}\,\frac{(s-1)-(s-3)}{(s-3)(s-1)}$$

$$= \frac{1}{(s-3)(s-1)} = F(s)\,G(s).$$

General Properties of Convolution

$$f * 0 = 0 * f = 0,$$

$$f * g = g * f,$$

$$(f * g) * h = f * (g * h)$$

$$f * (c_1 g_1 + c_2 g_2) = c_1 f * g_1 + c_2 f * g_2$$

Warning

- $1 * f \neq f$, in general
- $\delta(t) * f(t) = f(t)$
- $(f * g)(t) \neq f(t) g(t)$, in general