Transfer Function.
Impulse Response.

Example. Solve a General Nonhomogeneous Linear Problem

$$(*) \quad 2y''(t) + 9y'(t) + 4y(t) = f(t), \qquad y(0) = y_0, \qquad y'(0) = y_1.$$

Solution:

Impulse Response Problem $\quad 2h''(t) + 9h(t) + 4h(t) = \delta(t), \;\; h(0) = 0, \;\; h'(0) = 0.$

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Solution in the s-domain [**Transfer Function**]: $\quad 2s^2H(s) + 9sH(s) + 4H(s) = 1$

$$\Rightarrow H(s) = \frac{1}{2s^2+9s+4} = \frac{1}{(2s+1)(s+4)} = \frac{1/7}{s+1/2} + \frac{-1/7}{s+4}$$

Solution in the t-domain [**Impulse Response**]:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{7}e^{-\frac{t}{2}} - \frac{1}{7}e^{-4t}$$

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General Response Problem $(*)$

$$\xrightarrow{\mathcal{L}} \quad 2[s^2Y(s) - sy_0 - y_1] + 9[sY(s) - y_0] + 4Y(s) = F(s),$$

$$Y(s) = \frac{1}{2s^2+9s+4}[2y_1 + 9y_0 + 2sy_0 + F(s)] = (2y_1 + 9y_0)H(s) + 2y_0sH(s) + H(s)F(s).$$

$$\xrightarrow{\mathcal{L}^{-1}} \; y(t) = (2y_1 + 9y_0)h(t) + 2y_0h'(t) \quad + (h * f)(t) = y_c(t) \quad + y_p(t),$$ is the "**Total Response**"

$$y(t) = (2y_1 + 9y_0)\left[\frac{1}{7}e^{-\frac{t}{2}} - \frac{1}{7}e^{-4t}\right] + 2y_0\left[-\frac{1}{14}e^{-\frac{t}{2}} + \frac{4}{7}e^{-4t}\right] \quad + \int_0^t \left[\frac{1}{7}e^{-\frac{t-\tau}{2}} - \frac{1}{7}e^{-4(t-\tau)}\right]f(\tau)d\tau,$$

where $y_c(t) = (2y_1 + 9y_0)h(t) + 2y_0h'(t)$

is the "**Free Response**" satisfying $\quad 2y_c''(t) + 9y_c'(t) + 4y_c(t) = 0, \;\; y_c(0) = y_0, \;\; y_c'(0) = y_1,$

$y_p(t) = (h * f)(t)$

is the "**Forced Response**" satisfying $\quad 2y_p''(t) + 9y_p'(t) + 4y_p(t) = f(t), \;\; y_p(0) = 0, \;\; y_p'(0) = 0.$

Impulse Response Problem $a_2h''(t) + a_1h(t) + a_0h(t) = \delta(t), \;\; h(0) = 0, \;\; h'(0) = 0.$

Solution in the s-domain: $a_2s^2H(s) + a_1sH(s) + a_0H(s) = 1$

$\Rightarrow H(s) = \frac{1}{a_2s^2+a_1s+a_0}$ **[Transfer Function]**

Solution in the t-domain:

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ **[Impulse Response]**

General Response Problem $(*)$ $a_2y''(t) + a_1y(t) + a_0y(t) = f(t), \;\; y(0) = y_0, \;\; y'(0) = y_1.$

$\overset{\mathcal{L}}{\to} \;\; a_2[s^2Y(s) - sy_0 - y_1] + a_1[sY(s) - y_0] + a_0Y(s) = F(s),$

$$Y(s) = \frac{1}{a_2s^2+a_1s+a_0}[a_2y_1 + a_1y_0 + a_2y_0s + F(s)]$$
$$= (a_2y_1 + a_1y_0)H(s) + a_2y_0sH(s) + H(s)F(s).$$

$\overset{\mathcal{L}^{-1}}{\longrightarrow}$ "**Total Response**": $y(t) = (a_2y_1 + a_1y_0)h(t) + a_2y_0h'(t) + (h * f)(t) = y_c(t) + y_p(t),$

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"**Free Response**": $y_c(t) = (a_2y_1 + a_1y_0)h(t) + a_2y_0h'(t)$ satisfies

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"**Forced Response**": $y_p(t) = (h * f)(t) = \int_0^t h(t-\tau)f(\tau)d\tau$ satisfies

$$a_2y_p''(t) + a_1y_p'(t) + a_0y_p(t) = f(t), \;\; y_p(0) = 0, \;\; y_p'(0) = 0.$$

General Nonhomogeneous Problem (Total Response)

$$\begin{cases} a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \\ y(0) = y_0, \qquad y'(0) = y_1 \end{cases}$$

Solution in the s-domain: $Y(s) = Y_c(s) + Y_p(s)$

Solution in the t-domain: $y(t) = y_c(t) + y_p(t)$

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Impulse Response Problem

$$\begin{cases} a_2 h''(t) + a_1 h'(t) + a_0 h(t) = \delta(t) \\ h(0) = 0, \qquad h'(0) = 0 \end{cases}$$

Solution in the s-domain [**Transfer Function**]:

$$H(s) = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

Solution in the t-domain [**Impulse Response**]:

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Free Response Problem

$$\begin{cases} a_2 {y_c}''(t) + a_1 {y_c}'(t) + a_0 y_c(t) = 0 \\ y_c(0) = y_0, \qquad {y_c}'(0) = y_1 \end{cases}$$

Solution in the s-domain:

$$Y_c(s) = (a_2 y_1 + a_1 y_0) H(s) + a_2 y_0 s H(s)$$

Solution in the t-domain [**Free Response**]:

$$y_c(t) = (a_2 y_1 + a_1 y_0) h(t) + a_2 y_0 h'(t)$$

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$$\begin{cases} a_2 {y_p}''(t) + a_1 {y_p}'(t) + a_0 y_p(t) = f(t) \\ y_p(0) = 0, \qquad {y_p}'(0) = 0 \end{cases}$$

Solution in the s-domain:

$$Y_p(s) = H(s) F(s)$$

Solution in the t-domain [**Forced Response**]:

$$y_p(t) = (h * f)(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

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$$(*) \qquad 4y''(t) + 12y'(t) + 25y(t) = f(t),\ y(0) = y_0,\ y'(0) = y_1.$$

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Solution:

Impulse Response Problem

$\xrightarrow{\mathcal{L}}$

[Transfer Function] $H(s)$

$\xrightarrow{\mathcal{L}^{-1}}$ **[Impulse Response]** $h(t)$

General Response Problem $(*)$

$\xrightarrow{\mathcal{L}}$

$Y(s)$ expressed in terms of $H(s)$

$\xrightarrow{\mathcal{L}^{-1}}$ $y(t)$ expressed in terms of $h(t)$

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$$(*) \qquad 4y''(t) + 12y'(t) + 25y(t) = f(t),\ y(0) = y_0,\ y'(0) = y_1.$$

Solution:

Impulse Response Problem $\quad 4h''(t) + 12h(t) + 25h(t) = \delta(t),\ \ h(0) = 0,\ \ h'(0) = 0.$

$$\xrightarrow{\mathcal{L}} \qquad 4s^2H(s) + 12sH(s) + 25H(s) = 1$$

[Transfer Function] $\quad H(s) = \frac{1}{4s^2+12s+25} = \frac{1}{4(s^2+3s+\frac{25}{4})} = \frac{1}{4} \cdot \frac{1}{\left(s+\frac{3}{2}\right)^2+4}$

$\xrightarrow{\mathcal{L}^{-1}}$ **[Impulse Response]** $\quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{8}e^{-\frac{3t}{2}}\sin(2t)$

General Response Problem $(*)$

$\xrightarrow{\mathcal{L}}$

$Y(s)$ expressed in terms of $H(s)$

$\xrightarrow{\mathcal{L}^{-1}}$ $y(t)$ expressed in terms of $h(t)$

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[Transfer Function] $H(s) = \frac{1}{4s^2+12s+25} = \frac{1}{4(s^2+3s+\frac{25}{4})} = \frac{1}{4}\cdot\frac{1}{\left(s+\frac{3}{2}\right)^2+4}$

$\xrightarrow{\mathcal{L}^{-1}}$ **[Impulse Response]** $h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{8}e^{-\frac{3t}{2}}\sin(2t)$

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$$\xrightarrow{\mathcal{L}^{-1}} y(t) = (4y_1 + 12y_0)h(t) + 4y_0h'(t) \quad + (h * f)(t) = y_c(t) \quad + y_p(t)$$

$$= (4y_1 + 12y_0)\frac{1}{8}e^{-\frac{3t}{2}}\sin(2t) + 4y_0\left[-\frac{3}{16}e^{-\frac{3}{2}t}\sin(2t) + \frac{1}{4}e^{-\frac{3}{2}t}\cos(2t)\right]$$

$$+ \int_0^t \frac{1}{8}e^{-\frac{3(t-\tau)}{2}}\sin 2(t-\tau)\, f(\tau)d\tau\,,$$

Example. Find a particular solution of

$$(*) \qquad y''(t) - 2y'(t) + y(t) = \frac{e^t}{t^2+1}.$$

Solution 1: Variation of Parameters

Solution 2: Use $\mathcal{L}$, $\mathcal{L}^{-1}$ and Convolution

Transfer Function $H(s)$

Impulse Response $h(t) = \mathcal{L}^{-1}\{H(s)\}$

The Forced Response is a particular solution of the nonhomogeneous equation:

$$y_p(t) = (h * f)(t) = \int_0^t h(t-\tau)\, f(\tau)\, d\tau$$

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$$= \int_0^t \frac{(t-\tau)e^t}{\tau^2+1} d\tau = \int_0^t \left(\frac{te^t}{\tau^2+1} - \frac{\tau e^t}{\tau^2+1}\right) d\tau$$

$$= \int_0^t \left(\frac{te^t}{\tau^2+1} - \frac{\tau e^t}{\tau^2+1}\right) d\tau = \left[te^t \arctan\tau - \frac{1}{2}e^t \ln(\tau^2+1)\right]_{\tau=0}^{\tau=t}$$

$$= \boxed{te^t \arctan t - \frac{1}{2}e^t \ln(t^2+1)}$$

Example. Find a particular solution of

$$(*) \qquad y''(t) - y(t) = \frac{1}{\cosh(t)}.$$

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The Forced Response is a particular solution of the nonhomogeneous equation:

$$y_p(t) = (h * f)(t) = \int_0^t h(t-\tau)\, f(\tau) d\tau = \int_0^t \sinh(t-\tau)\, \frac{1}{\cosh(\tau)} d\tau$$

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The Forced Response is a particular solution of the nonhomogeneous equation:

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$$= \int_0^t [\sinh(t)\cosh(\tau) - \cosh(t)\sinh(\tau)]\, \frac{1}{\cosh(\tau)} d\tau$$

$$= \int_0^t \left[\sinh(t) - \cosh(t)\, \frac{\sinh(\tau)}{\cosh(\tau)}\right] d\tau$$

Substitute $v = \cosh(\tau)$.
We have $dv = \sinh(\tau) d\tau$.

$$= t\sinh(t) - \cosh(t)\, \left[\ln(\cosh \tau)\right]_{\tau=0}^{\tau=t}$$

$\cosh(0) = \frac{e^0 + e^{-0}}{2} = 1.$

$$= \boxed{t\sinh(t) - \cosh(t)\ln(\cosh t)}$$