Examples of Ordinary Diff Equations

Solved by the Laplace Transform

Examples 1-2 : First order diff eq's

Examples 3-6 : Second order diff eq's

Example 7 : A Fourth order diff eq

Examples 8-9 : Systems of diff eq's

Example 1 Solve $y'(t) + 3y(t) = 5e^{-7t}, \quad y(0) = 9.$

Solution

$$\Big\downarrow \mathcal{L}$$

$$[sY(s) - 9] + 3Y(s) = \frac{5}{s+7}. \quad \Rightarrow Y(s) = \frac{9}{s+3} + \frac{5}{(s+3)(s+7)}$$

$$= (s+3)Y(s) - 9$$

Partial Fraction Set $Y(s) = \dfrac{9}{s+3} + \dfrac{5}{(s+3)(s+7)} = \dfrac{a}{s+3} + \dfrac{b}{s+7}.$

Multiply both sides by $(s+3)(s+7)$:

$$(*) \quad 9(s+7) + 5 = a(s+7) + b(s+3)$$

Let $s = -3$ in $(*)$: $\quad 9(4) + 5 = a(4) \Rightarrow a = \dfrac{41}{4}$

$s = -7$: $\quad 5 = b(-4) \Rightarrow b = -\dfrac{5}{4}$

$$\left.\begin{array}{l} a = \frac{41}{4} \\ b = -\frac{5}{4} \end{array}\right\} \; Y(s) = \frac{41/4}{s+3} + \frac{-5/4}{s+7}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{y(t) = \frac{41}{4}e^{-3t} - \frac{5}{4}e^{-7t}}$$

Example 2 Solve $y'(t) + 3y(t) = 4te^{-7t}$, $y(0) = 7$.

Solution $\xrightarrow{\mathcal{L}}$ $sY(s) - 7 + 3Y(s) = \dfrac{4}{(s+7)^2}$.

$$Y(s) = \frac{7}{s+3} + \frac{4}{(s+3)(s+7)^2}.$$

Partial Fraction for the 2nd Term: Set $\dfrac{4}{(s+3)(s+7)^2} = \dfrac{A}{s+3} + \dfrac{B}{s+7} + \dfrac{C}{(s+7)^2}$

Multiply both sides by $(s+3)(s+7)^2$: $4 = A(s+7)^2 + B(s+3)(s+7) + C(s+3)$ (*)

Let $s = -3$ in (*): $4 = A(4)^2 \Rightarrow A = \frac{1}{4}$.

$s = -7$ in (*): $4 = C(-4) \Rightarrow C = -1$.

$s = 0$ in (*): $4 = A(7)^2 + B(3)(7) + C(3) = \frac{49}{4} + 21B - 3 \Rightarrow B = -\frac{1}{4}$.

$$Y(s) = \frac{7}{s+3} + \frac{\frac{1}{4}}{s+3} + \frac{-\frac{1}{4}}{s+7} + \frac{-1}{(s+7)^2} = \frac{\frac{29}{4}}{s+3} + \frac{-\frac{1}{4}}{s+7} + \frac{-1}{(s+7)^2}$$

$\xrightarrow{\mathcal{L}^{-1}}$ $\boxed{y(t) = \frac{29}{4}e^{-3t} - \frac{1}{4}e^{-7t} - te^{-7t}}$

Example 3 $\begin{cases} y''(t) + 2y'(t) - 8y(t) = e^{3t} \\ y(0) = 6, \quad y'(0) = -2 \end{cases}$

Solution $\downarrow \mathcal{L}$

- $[s^2 Y(s) - 6s - (-2)] + 2[sY(s) - 6] - 8Y(s) = \dfrac{1}{s-3}$,

$(s^2+2s-8)Y(s) - 6s - 10 = \dfrac{1}{s-3}$, $\quad Y(s) = \dfrac{6s+10}{s^2+2s-8} + \dfrac{1}{(s^2+2s-8)(s-3)}$.

- Factorize $s^2+2s-8 = (s-2)(s+4)$.
- Set $Y(s) = \dfrac{6s+10}{(s-2)(s+4)} + \dfrac{1}{(s-2)(s+4)(s-3)} = \dfrac{A}{s-2} + \dfrac{B}{s+4} + \dfrac{C}{s-3}$.
- Multiply both sides by $(s-2)(s+4)(s-3)$:

$(*)\quad (6s+10)(s-3) + 1 = A(s+4)(s-3) + B(s-2)(s-3) + C(s-2)(s+4)$.

Let $s = 2$ in $(*)$: $(22)(-1)+1 = A(6)(-1) \Rightarrow A = \frac{7}{2}$.

$s = -4$: $(-14)(-7)+1 = B(-6)(-7) \Rightarrow B = \frac{33}{14}$.

$s = 3$: $1 = C(1)(7) \Rightarrow C = \frac{1}{7}$.

- $Y(s) = \dfrac{7/2}{s-2} + \dfrac{33/14}{s+4} + \dfrac{1/7}{s-3}$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{7}{2}e^{2t} + \frac{33}{14}e^{-4t} + \frac{1}{7}e^{3t}}$$

Example 4 Solve $\begin{cases} y''(t)+2y'(t)-8y(t) = \cos(3t)+8\sin(3t) \\ y(0)=6, \quad y'(0)=-2 \end{cases}$

Solution: $\mathcal{L} \downarrow$

- $$[s^2 Y(s) - 6s - (-2)] + 2[sY(s) - 6] - 8Y(s) = \frac{s}{s^2+9} + \frac{8\cdot 3}{s^2+9},$$

$$(s^2+2s-8)Y(s) - 6s - 10 = \frac{s+24}{s^2+9}, \quad Y(s) = \frac{6s+10}{s^2+2s-8} + \frac{s+24}{(s^2+2s-8)(s^2+9)}.$$

- Factorize $s^2+2s-8 = (s-2)(s+4)$

Partial Fraction

- $$Y(s) = \frac{6s+10}{(s-2)(s+4)} + \frac{s+24}{(s-2)(s+4)(s^2+9)} \overset{\downarrow}{=} \frac{A}{s-2} + \frac{B}{s+4} + \frac{Cs+D}{s^2+9}$$

$$(*) \quad (6s+10)(s^2+9) + s + 24 = A(s+4)(s^2+9) + B(s-2)(s^2+9) + (Cs+D)(s-2)(s+4).$$

$s=2$ in $(*)$: $(22)(13)+2+24 = A(6)(13) \Rightarrow A = 4$

$s=-4$ in $(*)$: $(-14)(25)-4+24 = B(-6)(25) \Rightarrow B = \frac{11}{5}$

$s=0$ in $(*)$: $(10)(9)+24 = \underset{4}{\cancel{A}}(4)(9) + \underset{11/5}{\cancel{B}}(-2)(9) + D(-2)(4) \Rightarrow D = -\frac{6}{5}$

$s=1$ in $(*)$: $(16)(10)+1+24 = \underset{4}{\cancel{A}}(5)(10) + \underset{11/5}{\cancel{B}}(-1)(10) + (C+\underset{-\frac{6}{5}}{\cancel{D}})(-1)(5) \Rightarrow C = -\frac{1}{5}$

- $$Y(s) = \frac{4}{s-2} + \frac{11/5}{s+4} + \frac{-\frac{1}{5}s - \frac{6}{5}}{s^2+9}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = 4e^{2t} + \frac{11}{5}e^{-4t} - \frac{1}{5}\cos(3t) - \frac{2}{5}\sin(3t)}$$

Example 4 Solve $\begin{cases} y''(t)+2y'(t)-8y(t) = \cos(3t)+8\sin(3t) \\ y(0)=6, \quad y'(0)=-2 \end{cases}$

Solution: $\mathcal{L} \downarrow$

- $[s^2Y(s)-6s-(-2)]+2[sY(s)-6]-8Y(s) = \dfrac{s}{s^2+9}+\dfrac{8\cdot 3}{s^2+9}$,

$(s^2+2s-8)Y(s)-6s-10 = \dfrac{s+24}{s^2+9}$, $\quad Y(s) = \dfrac{6s+10}{s^2+2s-8}+\dfrac{s+24}{(s^2+2s-8)(s^2+9)}$.

- Factorize $s^2+2s-8 = (s-2)(s+4)$

Partial Fraction

- $Y(s) = \dfrac{6s+10}{(s-2)(s+4)}+\dfrac{s+24}{(s-2)(s+4)(s^2+9)} = \dfrac{A}{s-2}+\dfrac{B}{s+4}+\dfrac{Cs+D}{s^2+9}$

(*) $(6s+10)(s^2+9)+s+24 = A(s+4)(s^2+9)+B(s-2)(s^2+9)+(Cs+D)(s-2)(s+4)$.

$s=2$ in (*): $\quad (22)(13)+2+24 = A(6)(13) \Rightarrow A=4$

$s=-4$ in (*): $\quad (-14)(25)-4+24 = B(-6)(25) \Rightarrow B=\frac{11}{5}$

Expand both sides of (*) and compare the coefficients of s^3 and s^0:

coefficients of s^3: $\quad 6 = A+B+C = 4+\frac{11}{5}+C \Rightarrow C=-\frac{1}{5}$

coefficients of s^0: $\quad (10)(9)+24 = \cancel{A}^{4}(4)(9)+\cancel{B}^{11/5}(-2)(9)+D(-2)(4) \Rightarrow D=-\frac{6}{5}$

Another Way of getting C & D

- $Y(s) = \dfrac{4}{s-2}+\dfrac{11/5}{s+4}+\dfrac{-\frac{1}{5}s-\frac{6}{5}}{s^2+9}$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = 4e^{2t}+\frac{11}{5}e^{-4t}-\frac{1}{5}\cos(3t)-\frac{2}{5}\sin(3t)}$$

Example 4 Solve $\begin{cases} y''(t)+2y'(t)-8y(t) = \cos(3t)+8\sin(3t) \\ y(0)=6, \quad y'(0)=-2 \end{cases}$

Solution : $\mathcal{L}\downarrow$

- $[s^2Y(s)-6s-(-2)]+2[sY(s)-6]-8Y(s) = \dfrac{s}{s^2+9}+\dfrac{8\cdot 3}{s^2+9}$,

$(s^2+2s-8)Y(s)-6s-10 = \dfrac{s+24}{s^2+9}$, $\quad Y(s) = \dfrac{6s+10}{s^2+2s-8}+\dfrac{s+24}{(s^2+2s-8)(s^2+9)}$.

- Factorize $s^2+2s-8 = (s-2)(s+4)$

Partial Fraction

- $Y(s) = \dfrac{6s+10}{(s-2)(s+4)}+\dfrac{s+24}{(s-2)(s+4)(s^2+9)} \overset{\downarrow}{=} \dfrac{A}{s-2}+\dfrac{B}{s+4}+\dfrac{Cs+D}{s^2+9}$

(*) $(6s+10)(s^2+9)+s+24 = A(s+4)(s^2+9)+B(s-2)(s^2+9)+(Cs+D)(s-2)(s+4)$.

$s=2$ in (*): $\quad (22)(13)+2+24 = A(6)(13) \Rightarrow A=4$

$s=-4$ in (*): $\quad (-14)(25)-4+24 = B(-6)(25) \Rightarrow B=\frac{11}{5}$

$s=3i$ in (*): $\quad 3i+24 = (3Ci+D)(3i-2)(3i+4)$,

$3Ci+D = \dfrac{3i+24}{(3i-2)(3i+4)} = \cdots$ Simplify $\cdots = -\dfrac{6}{5}-\dfrac{3}{5}i$

$\Rightarrow C=-\frac{1}{5}$, $D=-\frac{6}{5}$

3rd way of getting C & D

- $Y(s) = \dfrac{4}{s-2}+\dfrac{11/5}{s+4}+\dfrac{-\frac{1}{5}s-\frac{6}{5}}{s^2+9}$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = 4e^{2t}+\frac{11}{5}e^{-4t}-\frac{1}{5}\cos(3t)-\frac{2}{5}\sin(3t)}$$

Example[5] Solve $y''(t)+2y'(t)+5y(t)=10\cos t$, $y(0)=-4$, $y'(0)=5$.

Solution $\xrightarrow{\mathcal{L}}$ $[s^2Y(s)-(-4)s-5]+2[sY(s)-(-4)]+5Y(s)=\dfrac{10s}{s^2+1}$.

$$=(s^2+2s+5)Y(s)+4s+3$$

$$Y(s)=\frac{-4s-3}{s^2+2s+5}+\frac{10s}{(s^2+2s+5)(s^2+1)}$$

Partial Fraction for the 2nd fraction $\dfrac{10s}{(s^2+2s+5)(s^2+1)}=\dfrac{A_1s+A_0}{s^2+1}+\dfrac{B_1s+B_0}{s^2+2s+5}$

$$(*)\quad 10s=(A_1s+A_0)(s^2+2s+5)+(B_1s+B_0)(s^2+1)$$

$s=0$ in $(*)$: 1 $\quad 5A_0+B_0=0$

$s=1$ in $(*)$: 2 $\quad 8(A_1+A_0)+2(B_1+B_0)=10$

$s=-1$ in $(*)$: 3 $\quad 4(-A_1+A_0)+2(-B_1+B_0)=-10$

Coeff of s^3: 4 $\quad A_1+B_1=0$

Plug 1 4 in 2 3 $\Rightarrow$ 5 $\; 6A_1-2A_0=10$, 6 $\; -2A_1-6A_0=-10$

5 + 3 6 : $-20A_0=-20$, $A_0=1$,

3 5 − 6 : $20A_1=40$, $A_1=2$.

use 1 4 : $B_0=-5A_0=-5$, $B_1=-A_1=-2$.

Method 1 of getting A_1, A_0, B_1, B_0

$$Y(s)=\frac{-4s-3}{s^2+2s+5}+\frac{2s+1}{s^2+1}+\frac{-2s-5}{s^2+2s+5}=\frac{2s+1}{s^2+1}+\frac{-6s-8}{s^2+2s+5}=\frac{2s+1}{s^2+1}+\frac{-6(s+1)-2}{(s+1)^2+4}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t)=2\cos t+\sin t+e^{-t}\mathcal{L}^{-1}\left\{\frac{-6s-2}{s^2+4}\right\}=2\cos t+\sin t+e^{-t}(-6\cos 2t-\sin 2t)}$$

Example 5 Solve $y''(t)+2y'(t)+5y(t)=10\cos t,\quad y(0)=-4,\quad y'(0)=5.$

Solution $\xrightarrow{\mathcal{L}}$ $[s^2Y(s)-(-4)s-5]+2[sY(s)-(-4)]+5Y(s)=\dfrac{10s}{s^2+1}.$

$$=(s^2+2s+5)Y(s)+4s+3$$

$$Y(s)=\frac{-4s-3}{s^2+2s+5}+\frac{10s}{(s^2+2s+5)(s^2+1)}$$

Partial Fraction for the 2nd fraction $\dfrac{10s}{(s^2+2s+5)(s^2+1)}=\dfrac{A_1s+A_0}{s^2+1}+\dfrac{B_1s+B_0}{s^2+2s+5}$

$(*)\quad 10s=(A_1s+A_0)(s^2+2s+5)+(B_1s+B_0)(s^2+1)$

Method 2 of getting A_1, A_0, B_1, B_0

$s=i$ in $(*)$: $10i=(A_1i+A_0)(-1+2i+5)=(A_1i+A_0)(4+2i)$

$\Rightarrow\quad A_1i+A_0=\dfrac{10i}{4+2i}=\dfrac{5i}{2+i}=2i+1 \quad\Rightarrow\quad A_1=2,\quad A_0=1$

$s=-1+2i$ in $(*)$: $10(-1+2i)=[B_1(-1+2i)+B_0](1-4i-4+1)=[(B_0-B_1)+2B_1i](-2-4i)$

$\Rightarrow\ (B_0-B_1)+2B_1i=\dfrac{10(-1+2i)}{-2-4i}=-3-4i\ \Rightarrow\begin{cases}B_0-B_1=-3\\2B_1=-4\end{cases}\ \Rightarrow\begin{cases}B_0=-5\\B_1=-2\end{cases}$

$$Y(s)=\frac{-4s-3}{s^2+2s+5}+\frac{2s+1}{s^2+1}+\frac{-2s-5}{s^2+2s+5}=\frac{2s+1}{s^2+1}+\frac{-6s-8}{s^2+2s+5}=\frac{2s+1}{s^2+1}+\frac{-6(s+1)-2}{(s+1)^2+4}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t)=2\cos t+\sin t+e^{-t}\mathcal{L}^{-1}\left\{\frac{-6s-2}{s^2+4}\right\}=2\cos t+\sin t+e^{-t}(-6\cos 2t-\sin 2t)}$$

Example 6. Solve $y''(t) + y(t) = 15\sin(2t)$, $y(0) = 3$, $y'(0) = 8$.

Solution $\downarrow \mathcal{L}$

$$[s^2 Y(s) - 3s - 8] + Y(s) = 15 \cdot \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 3s - 8 = \frac{30}{s^2+4}$$

$$Y(s) = \frac{3s+8}{s^2+1} + \frac{30}{(s^2+1)(s^2+4)}$$

Partial Fraction for the 2nd fraction term: $\dfrac{30}{(s^2+1)(s^2+4)} = 10\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right)$

$$Y(s) = \frac{3s+8}{s^2+1} + \frac{10}{s^2+1} - \frac{10}{s^2+4} = \frac{3s+18}{s^2+1} - \frac{10}{s^2+4}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = 3\cos t + 18\sin t - 5\sin(2t)}$$

Example 7. Solve $\frac{d^4y}{dt^4} - y = 2e^{-t}$, $y(0)=4$, $y'(0)=3$, $y''(0)=-1$, $y'''(0)=-2$.

Solution $\xrightarrow{\mathcal{L}}$ $s^4 Y(s) - 4s^3 - 3s^2 - (-1)s - (-2) - Y(s) = \frac{2}{s+1}$.

- $Y(s) = \frac{4s^3+3s^2-s-2}{s^4-1} + \frac{2}{(s^4-1)(s+1)}$.   factorize $s^4-1 = (s^2-1)(s^2+1) = (s-1)(s+1)(s^2+1)$

$$= \frac{4s^3+3s^2-s-2}{(s-1)(s+1)(s^2+1)} + \frac{2}{(s-1)(s+1)^2(s^2+1)} = \frac{A}{s-1} + \frac{B_1}{s+1} + \frac{B_2}{(s+1)^2} + \frac{C_1 s + C_2}{s^2+1}.$$

- (*) $(4s^3+3s^2-s-2)(s+1) + 2 = A(s+1)^2(s^2+1) + B_1(s-1)(s+1)(s^2+1) + B_2(s-1)(s^2+1)$
$+ (C_1 s + C_2)(s-1)(s+1)^2$.

- Let $s=1$ in (*): $(4)(2)+2 = A(2)^2(2) \Rightarrow A = 5/4$.
  $s=-1$ in (*): $2 = B_2(-2)(2) \Rightarrow B_2 = -1/2$.
  $s=i$ in (*): $(-5-5i)(i+1) + 2 = (C_1 i + C_2)(i-1)(i+1)^2 = -2(1+i)(C_1 i + C_2)$
  $\Rightarrow C_1 i + C_2 = \frac{2-10i}{-2(1+i)} = 2+3i \Rightarrow C_1 = 3,\ C_2 = 2$.
  $s=0$ in (*): $(-2)(1)+2 = A - B_1 - B_2 - C_2 \Rightarrow 0 = \frac{5}{4} - B_1 - (-\frac{1}{2}) - 2 \Rightarrow B_1 = -1/4$.

- $Y(s) = \frac{5/4}{s-1} + \frac{-1/4}{s+1} + \frac{-1/2}{(s+1)^2} + \frac{3s+2}{s^2+1}$.

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{5}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-t} + 3\cos t + 2\sin t}$$

Example[8] Solve $\frac{d\mathbf{x}}{dt}(t) = A\,\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, where $A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$

Solution Let $\mathbf{X}(s) = \mathcal{L}\{\mathbf{x}(t)\}$, i.e. $\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \mathcal{L}\{x_2(t)\} \end{bmatrix}$

$$\xrightarrow{\mathcal{L}} \quad s\mathbf{X}(s) - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = A\mathbf{X}(s)$$

$$(sI - A)\mathbf{X}(s) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\mathbf{X}(s) = (sI-A)^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} s+3 & -2 \\ -1 & s+4 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{s^2+7s+10}\begin{bmatrix} s+4 & 2 \\ 1 & s+3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2s+14}{s^2+7s+10} \\[2ex] \dfrac{3s+11}{s^2+7s+10} \end{bmatrix} = \begin{bmatrix} \dfrac{2s+14}{(s+2)(s+5)} \\[2ex] \dfrac{3s+11}{(s+2)(s+5)} \end{bmatrix}$$

Partial Fractions $\Rightarrow$ $\mathbf{X}(s) = \begin{bmatrix} \dfrac{10/3}{s+2} + \dfrac{-4/3}{s+5} \\[2ex] \dfrac{5/3}{s+2} + \dfrac{4/3}{s+5} \end{bmatrix}$

$$\xleftarrow{\mathcal{L}^{-1}}$$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{10}{3}e^{-2t} - \frac{4}{3}e^{-5t} \\[1ex] \frac{5}{3}e^{-2t} + \frac{4}{3}e^{-5t} \end{bmatrix}$$

Example 9 Solve $\frac{d\mathbf{x}}{dt}(t) = \begin{bmatrix} -5 & 20 \\ -2 & 7 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Solution $\xrightarrow{\mathcal{L}}$ $s\mathbf{X}(s) - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & 20 \\ -2 & 7 \end{bmatrix} \mathbf{X}(s) + \begin{bmatrix} \frac{1}{s+3} \\ -\frac{2}{s+3} \end{bmatrix}$

$$\left(sI - \begin{bmatrix} -5 & 20 \\ -2 & 7 \end{bmatrix}\right) \mathbf{X}(s) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\mathbf{X}(s) = \left(sI - \begin{bmatrix} -5 & 20 \\ -2 & 7 \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) = \frac{1}{s^2-2s+5} \begin{bmatrix} s-7 & 20 \\ -2 & s+5 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

$$= \frac{1}{s^2-2s+5} \begin{bmatrix} 4s+12 \\ 2s+2 \end{bmatrix} + \frac{1}{(s^2-2s+5)(s+3)} \begin{bmatrix} s-47 \\ -2s-12 \end{bmatrix}$$

Partial Fraction for the 2nd Term (details are on the next page)

$$= \frac{1}{s^2-2s+5} \begin{bmatrix} 4s+12 \\ 2s+2 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix} + \frac{1}{s^2-2s+5} \left(s \begin{bmatrix} 5/2 \\ 3/10 \end{bmatrix} + \begin{bmatrix} -23/2 \\ -7/2 \end{bmatrix} \right)$$

$$= \frac{1}{s^2-2s+5} \begin{bmatrix} \frac{13}{2}s + \frac{1}{2} \\ \frac{23}{10}s - \frac{3}{2} \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix} = \frac{1}{(s-1)^2+4} \begin{bmatrix} \frac{13}{2}(s-1) + 7 \\ \frac{23}{10}(s-1) + \frac{4}{5} \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix},$$

$\xrightarrow{\mathcal{L}^{-1}}$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{13}{2} e^{t} \cos(2t) + \frac{7}{2} e^{t} \sin(2t) \\ \frac{23}{10} e^{t} \cos(2t) + \frac{2}{5} e^{t} \sin(2t) \end{bmatrix} + \begin{bmatrix} -\frac{5}{2} e^{-3t} \\ -\frac{3}{10} e^{-3t} \end{bmatrix}$$

Details of
the Partial Fractions for the 2nd Term:

Set
$$\frac{1}{(s^2-2s+5)(s+3)}\begin{bmatrix} s-47 \\ -2s-12 \end{bmatrix} = \frac{1}{s+3}\vec{a} + \frac{1}{s^2-2s+5}(s\vec{b}+\vec{c}).$$

$$\begin{bmatrix} s-47 \\ -2s-12 \end{bmatrix} = (s^2-2s+5)\vec{a} + (s+3)(s\vec{b}+\vec{c}).$$

Let $s=-3$: $\begin{bmatrix} -50 \\ -6 \end{bmatrix} = 20\vec{a}$, $\vec{a} = \begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix}$

$s=0$: $\begin{bmatrix} -47 \\ -12 \end{bmatrix} = 5\vec{a}+3\vec{c} = \begin{bmatrix} -25/2 \\ -3/2 \end{bmatrix} + 3\vec{c}$, $\vec{c} = \begin{bmatrix} -23/2 \\ -7/2 \end{bmatrix}$

$s=1$: $\begin{bmatrix} -46 \\ -14 \end{bmatrix} = 4\vec{a}+4\vec{b}+4\vec{c} = 4\begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix} + 4\vec{b} + 4\begin{bmatrix} -23/2 \\ -7/2 \end{bmatrix}$. $\vec{b} = \begin{bmatrix} 5/2 \\ 3/10 \end{bmatrix}$

Thus,
$$\frac{1}{(s^2-2s+5)(s+3)}\begin{bmatrix} s-47 \\ -2s-12 \end{bmatrix} = \frac{1}{s+3}\begin{bmatrix} -5/2 \\ -3/10 \end{bmatrix} + \frac{1}{s^2-2s+5}\left(\begin{bmatrix} 5/2 \\ 3/10 \end{bmatrix}s + \begin{bmatrix} -23/2 \\ -7/2 \end{bmatrix}\right)$$

Solution 2 of Example 9 (using complex exp functions) Partial Fraction for $\mathbf{X}(s)$:

$s^2-2s+5=0$
$\Rightarrow s=1\pm 2i$

$$\mathbf{X}(s)=\frac{1}{s^2-2s+5}\begin{bmatrix}4s+12\\2s+2\end{bmatrix}+\frac{1}{(s^2-2s+5)(s+3)}\begin{bmatrix}s-47\\-2s-12\end{bmatrix}$$

$$=\frac{1}{[s-(1+2i)][s-(1-2i)]}\begin{bmatrix}4s+12\\2s+2\end{bmatrix}+\frac{1}{[s-(1+2i)][s-(1-2i)](s+3)}\begin{bmatrix}s-47\\-2s-12\end{bmatrix}$$

$$=\frac{1}{s-(1+2i)}\vec{u}+\frac{1}{s-(1-2i)}\vec{v}+\frac{1}{s+3}\vec{w},$$

$$(**)\quad (s+3)\begin{bmatrix}4s+12\\2s+2\end{bmatrix}+\begin{bmatrix}s-47\\-2s-12\end{bmatrix}=[s-(1-2i)](s+3)\vec{u}+[s-(1+2i)](s+3)\vec{v}+(s^2-2s+5)\vec{w}.$$

Let $s=-3$ in $(**)$: $\begin{bmatrix}-50\\-6\end{bmatrix}=20\vec{w}\Rightarrow\vec{w}=\begin{bmatrix}-5/2\\-3/10\end{bmatrix}$

$s=1+2i$ in $(**)$: $(4+2i)\begin{bmatrix}16+8i\\4+4i\end{bmatrix}+\begin{bmatrix}-46+2i\\-14-4i\end{bmatrix}=(4i)(4+2i)\vec{u}\Rightarrow\vec{u}=\begin{bmatrix}\frac{13}{4}-\frac{7}{4}i\\\frac{23}{20}-\frac{1}{5}i\end{bmatrix}$

$\vec{v}=[\text{the complex conjugate of }\vec{u}]=\begin{bmatrix}\frac{13}{4}+\frac{7}{4}i\\\frac{23}{20}+\frac{1}{5}i\end{bmatrix}$

Thus, $$\mathbf{X}(s)=\frac{1}{s-(1+2i)}\begin{bmatrix}\frac{13}{4}-\frac{7}{4}i\\\frac{23}{20}-\frac{1}{5}i\end{bmatrix}+\frac{1}{s-(1-2i)}\begin{bmatrix}\frac{13}{4}+\frac{7}{4}i\\\frac{23}{20}+\frac{1}{5}i\end{bmatrix}+\frac{1}{s+3}\begin{bmatrix}-5/2\\-3/10\end{bmatrix}$$

$$\xrightarrow{\mathcal{L}^{-1}}\quad \mathbf{x}(t)=e^{(1+2i)t}\begin{bmatrix}\frac{13}{4}-\frac{7}{4}i\\\frac{23}{20}-\frac{1}{5}i\end{bmatrix}+e^{(1-2i)t}\begin{bmatrix}\frac{13}{4}+\frac{7}{4}i\\\frac{23}{20}+\frac{1}{5}i\end{bmatrix}+e^{-3t}\begin{bmatrix}-\frac{5}{2}\\-\frac{3}{10}\end{bmatrix}$$

$$x(t) = 2\,\mathrm{Re}\left\{ e^{(1+2i)t} \begin{bmatrix} \frac{13}{4} - \frac{7}{4} i \\ \frac{23}{20} - \frac{1}{5} i \end{bmatrix} \right\} + e^{-3t} \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{10} \end{bmatrix}$$

$$= 2\,\mathrm{Re}\left\{ e^{t} [\cos(2t) + i \sin(2t)] \begin{bmatrix} \frac{13}{4} - \frac{7}{4} i \\ \frac{23}{20} - \frac{1}{5} i \end{bmatrix} \right\} + e^{-3t} \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{10} \end{bmatrix}$$

$$= 2 e^{t} \cos(2t) \begin{bmatrix} \frac{13}{4} \\ \frac{23}{20} \end{bmatrix} + 2 e^{t} \sin(2t) \begin{bmatrix} \frac{7}{4} \\ \frac{1}{5} \end{bmatrix} + e^{-3t} \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{10} \end{bmatrix}$$

$$= e^{t} \cos(2t) \begin{bmatrix} \frac{13}{2} \\ \frac{23}{10} \end{bmatrix} + e^{t} \sin(2t) \begin{bmatrix} \frac{7}{2} \\ \frac{2}{5} \end{bmatrix} + e^{-3t} \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{10} \end{bmatrix}$$