The Laplace Transform and The Inverse Laplace Transform

— an introduction.

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$$

$$\cos(2t) \xrightarrow{\mathcal{L}} \frac{s}{s^2+4}$$

$$\frac{1}{s^4} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{6}t^3$$

$\boxed{\text{t - DOMAIN}}$

$$\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}$$

$\boxed{\text{s-DOMAIN}}$

$$sY(s) - 2 + 3Y(s) = \frac{6}{s-4}$$

$\boxed{\text{t- DOMAIN}}$

$$\boxed{\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}}$$

$$\xrightarrow{\mathcal{L}}$$

$\boxed{\text{s-DOMAIN}}$

$$\boxed{sY(s) - 2 + 3Y(s) = \frac{6}{s-4}}$$

$\boxed{t\text{- DOMAIN}}$

$$\boxed{\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}}$$

$$\xrightarrow{\mathcal{L}}$$

$\boxed{s\text{-DOMAIN}}$

$$\boxed{sY(s) - 2 + 3Y(s) = \frac{6}{s-4}}$$

$$Y(s) = \frac{2}{s+3} + \frac{6}{(s+3)(s-4)}$$

t - DOMAIN

$$\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}$$

$\xrightarrow{\mathcal{L}}$

s - DOMAIN

$$sY(s) - 2 + 3Y(s) = \frac{6}{s-4}$$

$$Y(s) = \frac{2}{s+3} + \frac{6}{(s+3)(s-4)}$$

$$= \frac{2}{s+3} + \frac{-6/7}{s+3} + \frac{6/7}{s-4}$$

$$= \frac{8/7}{s+3} + \frac{6/7}{s-4}$$

t - DOMAIN

$$\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}$$

$\xrightarrow{\mathcal{L}}$

s-DOMAIN

$$sY(s) - 2 + 3Y(s) = \frac{6}{s-4}$$

$$Y(s) = \frac{2}{s+3} + \frac{6}{(s+3)(s-4)}$$

$$= \frac{2}{s+3} + \frac{-6/7}{s+3} + \frac{6/7}{s-4}$$

$$= \frac{8/7}{s+3} + \frac{6/7}{s-4}$$

$\xleftarrow{\mathcal{L}^{-1}}$

$$y(t) = \frac{8}{7}e^{-3t} + \frac{6}{7}e^{4t}$$

Definition of the Laplace Transform $\mathcal{L}$

Given a function $f(t)$ on $[0, \infty)$

$$\downarrow \mathcal{L}$$

Transformed to a new function $F(s)$.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

We write :

$$F(s) = \mathcal{L}\{f(t)\}, \qquad f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Example

$f(t) = 1$.

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

$$= \lim_{T\to\infty} \int_0^T e^{-st} dt$$

$$= \lim_{T\to\infty} \left[-\frac{e^{-st}}{s} \right]_{t=0}^{t=T}$$

$$= \lim_{T\to\infty} \left(-\frac{e^{-sT}}{s} + \frac{1}{s} \right) = \frac{1}{s}.$$

provided $s > 0$. for real s

$\mathrm{Re}\, s > 0$, for complex s.

Example. $f(t) = e^{4t}$.

$$F(s) = \mathcal{L}\{e^{4t}\} = \int_0^\infty e^{-st} e^{4t}\, dt$$

$$= \int_0^\infty e^{(4-s)t}\, dt$$

$$= \lim_{T\to\infty} \left[\frac{e^{(4-s)t}}{4-s} \right]_{t=0}^{t=T}$$

$$= \lim_{T\to\infty} \left[\frac{e^{(4-s)T}}{4-s} - \frac{1}{4-s} \right] = -\frac{1}{4-s}$$

$$= \frac{1}{s-4} \qquad \text{provided } \operatorname{Re}(4-s) < 0$$

i.e. $\operatorname{Re} s > 4$.

In general,

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (\operatorname{Re} s > \operatorname{Re} a), \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Example. $f(t) = \cos(3t)$.

$$F(s) = \mathcal{L}\{\cos(3t)\} = \int_0^\infty e^{-st} \cos(3t)\, dt$$

$$= \mathcal{L}\left\{\frac{e^{3it} + e^{-3it}}{2}\right\}$$

$$= \frac{1}{2}\frac{1}{s-3i} + \frac{1}{2}\frac{1}{s+3i} = \frac{1}{2}\cdot\frac{2s}{(s-3i)(s+3i)}$$

$$= \frac{s}{s^2+9} \quad (\operatorname{Re} s > \operatorname{Re}(3i) = 0)$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

Example $\mathcal{L}^{-1}\left\{\dfrac{3s-2}{7s^2+6}\right\} = ?$

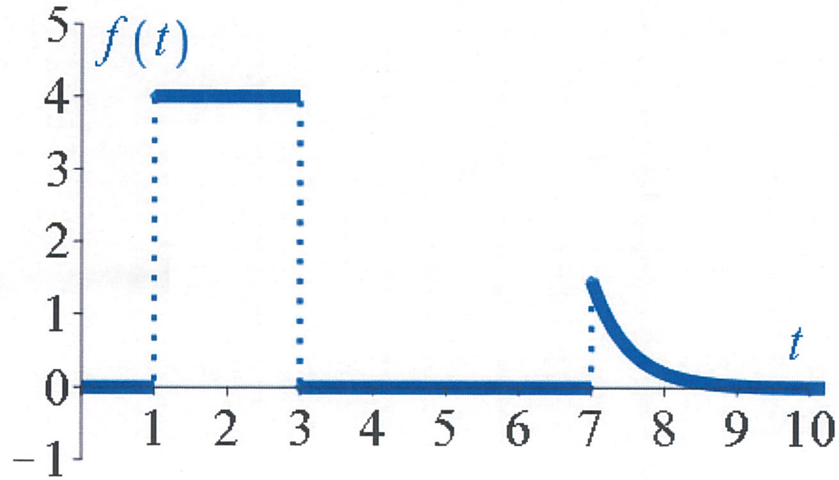
$$\frac{3s-2}{7s^2+6} = \frac{1}{7}\,\frac{3s-2}{s^2+\frac{6}{7}} = \frac{1}{7}\left(\frac{3s}{s^2+\frac{6}{7}} - \frac{2}{s^2+\frac{6}{7}}\right)$$

$$\xrightarrow[\mathcal{L}^{-1}]{} \frac{1}{7}\left[3\cos\left(\sqrt{\tfrac{6}{7}}\,t\right) - \frac{2}{\sqrt{\frac{6}{7}}}\sin\left(\sqrt{\tfrac{6}{7}}\,t\right)\right]$$

$$= \frac{3}{7}\cos\left(\sqrt{\tfrac{6}{7}}\,t\right) - \frac{2}{7}\sqrt{\frac{7}{6}}\,\sin\left(\sqrt{\tfrac{6}{7}}\,t\right).$$

Example

$$f(t) = \begin{cases} 0 & t<1 \\ 4 & 1 \le t < 3 \\ 0 & 3 \le t < 7 \\ 80e^{10-2t} & t \ge 7 \end{cases}$$



$F(s) = ?$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 0\,dt + \int_1^3 e^{-st} 4\,dt + \int_3^7 0\,dt + \int_7^\infty \underbrace{e^{-st}\, 80\, e^{10-2t}}_{80e^{10} e^{-(s+2)t}}\,dt$$

$$= \left[\frac{4e^{-st}}{-s}\right]_{t=1}^{t=3} + \left[80e^{10}\,\frac{e^{-(s+2)t}}{-(s+2)}\right]_{t=7}^{t=\infty}$$

$$= -\frac{4e^{-3s}}{s} + \frac{4e^{-s}}{s} + \frac{80e^{10}\,e^{-(s+2)7}}{s+2}$$

$$= -\frac{4e^{-3s}}{s} + \frac{4e^{-s}}{s} + \frac{80e^{-7s-4}}{s+2}.$$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

then $t f(t) \xrightarrow{\mathcal{L}} -\dfrac{dF}{ds}(s)$,

$$t^2 f(t) \xrightarrow{\mathcal{L}} \frac{d^2F}{ds^2}(s)$$

$$t^n f(t) \xrightarrow{\mathcal{L}} (-1)^n \frac{d^nF}{ds^n}(s)$$

$$\boxed{\text{Multiply by } t \text{ in the } t\text{-domain}} \quad \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \quad \boxed{\text{Apply } -\tfrac{d}{ds} \text{ in the } s\text{-domain}}$$

Why?

In $\int_0^\infty e^{-st} f(t)\,dt = F(s)$,

apply $\dfrac{d}{ds}$ on both sides:

$$\frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) = \frac{dF}{ds}(s)$$

$$\int_0^\infty e^{-st}(-t) f(t)\,dt = \frac{dF}{ds}(s) \Rightarrow \int_0^\infty e^{-st}\, t f(t)\,dt = -\frac{dF}{ds}(s)$$

$$= \mathcal{L}\{t f(t)\}$$

Example From $e^{7t} \xrightarrow{\mathcal{L}} \dfrac{1}{s-7}$

it follows that:

$$te^{7t} \xrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{1}{s-7}\right) = \frac{1}{(s-7)^2}$$

$$t^2e^{7t} \xrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{1}{(s-7)^2}\right) = \frac{2}{(s-7)^3}$$

$$t^3e^{7t} \xrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{2}{(s-7)^3}\right) = \frac{(2)(3)}{(s-7)^4} = \frac{6}{(s-7)^4}$$

$$t^ne^{7t} \xrightarrow{\mathcal{L}} \frac{n!}{(s-7)^{n+1}}$$

In general,

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad (\operatorname{Re} s > \operatorname{Re} a)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad (\operatorname{Re} s > 0)$$

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e.g.

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$$

$$\mathcal{L}^{-1}\left\{\frac{6}{s^6}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{5!}\,\frac{5!}{s^6}\right\} = \frac{6}{5!}t^5 = \frac{1}{20}t^5$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} = \frac{1}{3!}t^3 e^{-2t} = \frac{1}{6}t^3 e^{-2t}$$

Example.

From $\mathcal{L}\{\cos(3t)\} = \dfrac{s}{s^2+9}$, $\mathcal{L}\{\sin(3t)\} = \dfrac{3}{s^2+9}$

it follows :

$$\mathcal{L}\{t\cos(3t)\} = -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$$

$$= -\frac{(1)(s^2+9)-(s)(2s)}{(s^2+9)^2}$$

$$= \frac{s^2-9}{(s^2+9)^2},$$

$$\mathcal{L}\{t\sin(3t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -\frac{-3(2s)}{(s^2+9)^2}$$

$$= \frac{6s}{(s^2+9)^2}.$$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

then $e^{at} f(t) \xrightarrow{\mathcal{L}} F(s-a)$ and $F(s-a) \xrightarrow{\mathcal{L}^{-1}} e^{at} f(t)$

Why?

In $\int_0^\infty e^{-st} f(t)\, dt = F(s)$,

replace s by $s-7$:

$$\int_0^\infty e^{-(s-7)t} f(t)\, dt = F(s-7)$$

$$\int_0^\infty e^{-st} e^{7t} f(t)\, dt = F(s-7)$$

$$e^{7t} f(t) \xrightarrow{\mathcal{L}} F(s-7)$$

Example since $\cos(3t) \xrightarrow{\mathcal{L}} \dfrac{s}{s^2+9}$, $\sin(3t) \xrightarrow{\mathcal{L}} \dfrac{3}{s^2+9}$

we have $$e^{4t}\cos(3t) \xrightarrow{\mathcal{L}} \frac{s-4}{(s-4)^2+9}$$

$$e^{-t}\cos(3t) \xrightarrow{\mathcal{L}} \frac{s+1}{(s+1)^2+9}$$

$$e^{7t}\sin(3t) \xrightarrow{\mathcal{L}} \frac{3}{(s-7)^2+9}$$

$$e^{-6t}\sin(3t) \xrightarrow{\mathcal{L}} \frac{3}{(s+6)^2+9}$$

Example. $\mathcal{L}^{-1}\left\{\dfrac{4s+27}{s^2-4s+13}\right\} = ?$

$$\frac{4s+27}{s^2-4s+13} = \frac{4s+27}{(s-2)^2+9}$$

$$= \frac{4(s-2)+35}{(s-2)^2+9}$$

$$\downarrow \mathcal{L}^{-1}$$

$$e^{2t}\left[4\cos(3t) + \frac{35}{3}\sin(3t)\right]$$

Prepare:

$$\frac{4s+35}{s^2+9} = \frac{4s}{s^2+9} + \frac{35}{s^2+9}$$

$$\downarrow \mathcal{L}^{-1}$$

$$4\cos(3t) + \frac{35}{3}\sin(3t)$$

Definition (Exponential Order)

A function $f(t)$ is said to be of **exponential order** (as $t \to \infty$) if there exist a polynomial $p(t)$ and a real constant a such that $|f(t)| \le |p(t)| e^{at}$ for all large t.

Example $f(t) = 3e^{2t} + 5e^{7t}\cos(4t) - 9t^2 e^{6t}$ is of exp. order.

Since
$$
\begin{aligned}
|f(t)| &\le |3e^{2t}| + |5e^{7t}\cos(4t)| + |9t^2 e^{6t}| \\
&\le 3e^{2t} + 5e^{7t} + 9t^2 e^{6t} \\
&\le 3e^{7t} + 5e^{7t} + 9t^2 e^{7t} = (8 + 9t^2) e^{7t}.
\end{aligned}
$$

Non-Example $f(t) = e^{t^2}$ is NOT of exp. order,

since
$$\lim_{t\to\infty} \frac{f(t)}{|p(t)|e^{at}} = \lim_{t\to\infty} \frac{e^{t^2}}{|p(t)|e^{at}} = \lim_{t\to\infty} \frac{e^{t^2-at}}{|p(t)|} = \infty$$

for any polynomial $p(t)$ and any constant a.

Theorem (Existence of the Laplace Transform)

If
- $f(t)$ is piecewise continuous;
- $f(t)$ is of exponential order; that is,

$$|f(t)| \le |\text{a polynomial of } t|\, e^{at} \quad \text{for large } t,$$

then the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists for $\operatorname{Re} s > a$.

(or, for $s > a$, if we only consider real s)

Derivative Formulas

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

then $f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$

$$f''(t) \xrightarrow{\mathcal{L}} s^2F(s) - sf(0) - f'(0)$$

$$f'''(t) \xrightarrow{\mathcal{L}} s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$f^{(n)}(t) \xrightarrow{\mathcal{L}} s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)$$

$$- \cdots - f^{(n-1)}(0)$$

$$\boxed{f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)}$$

WHY?

$$(*) \qquad \int_0^\infty \frac{d}{dt}\left[e^{-st} f(t)\right] dt = \left[e^{-st} f(t)\right]_{t=0}^{t=\infty}$$

- L.H.S. of $(*)$ $= \int_0^\infty \left[(-se^{-st}) f(t) + e^{-st} f'(t)\right] dt$

$$= -s\int_0^\infty e^{-st} f(t)\, dt + \int_0^\infty e^{-st} f'(t)\, dt$$

$$= -s\,\mathcal{L}\{f(t)\} + \mathcal{L}\{f'(t)\}.$$

- R.H.S. of $(*)$ $= \lim_{t\to\infty} e^{-st} f(t) - e^0 f(0)$

$$= 0 - f(0)$$

$$= -f(0).$$

$\lim_{t\to\infty} e^{-st} f(t) = 0$ if $\begin{cases} f(t) \text{ is of exp. order} \\ \text{and } s \text{ is large enough} \end{cases}$

- $(*) \Rightarrow -sF(s) + \mathcal{L}\{f'(t)\} = -f(0)$

$$\Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

The first derivative formula $f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$

implies higher derivative formulas $f''(t) \xrightarrow{\mathcal{L}} s^2F(s) - sf(0) - f'(0)$

$\vdots$

- Denote $g(t) = f'(t)$.
- Use the 1st derivative formula for $f(t)$:

$$f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$$

i.e. $$g(t) \xrightarrow{\mathcal{L}} G(s) = sF(s) - f(0)$$

- Use the 1st derivative formula for $g(t)$:

$$f''(t) = g'(t) \xrightarrow{\mathcal{L}} sG(s) - g(0)$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Example $\begin{cases} y'(t) + 3y(t) = 6e^{4t} \\ y(0) = 2 \end{cases}$

$$\downarrow \mathcal{L}$$

the Eq for $Y(s)$

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use $\begin{cases} y(t) \xrightarrow{\mathcal{L}} Y(s) \\ y'(t) \xrightarrow{\mathcal{L}} sY(s) - y(0) \end{cases}$

$$\boxed{\left[sY(s) - 2\right] + 3Y(s) = \frac{6}{s-4}}$$

Example Transform $\begin{cases} y''(t) + 2y'(t) - 8y(t) = 7\sin(3t) \\ y(0) = 6, \quad y'(0) = -9 \end{cases}$

to an equation for $Y(s)$ in the s-domain.

Use $\begin{cases} y(t) \xrightarrow{\mathcal{L}} Y(s) \\ y'(t) \xrightarrow{\mathcal{L}} sY(s) - y(0) \\ y''(t) \xrightarrow{\mathcal{L}} s^2 Y(s) - s\,y(0) - y'(0) \end{cases}$

$$\boxed{\left[s^2 Y(s) - 6s + 9\right] + 2\left[sY(s) - 6\right] - 8Y(s) = \frac{21}{s^2+9}}$$