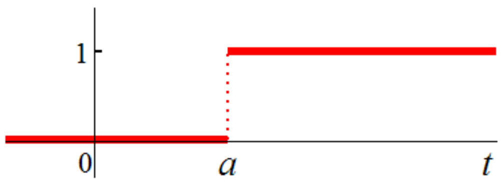
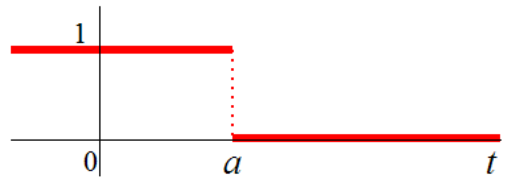
Laplace Transform of Piecewisely Defined Functions

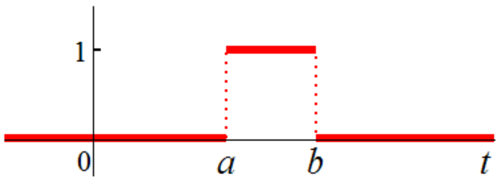
Unit Step Functions (of three types)



$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

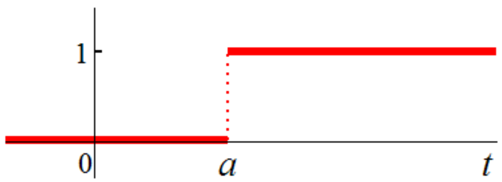


$$1 - u_a(t) = 1 - u(t-a) = \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$$

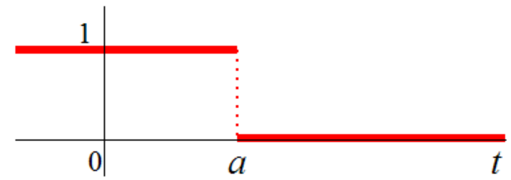


$$u_a(t) - u_b(t) = u(t-a) - u(t-b) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

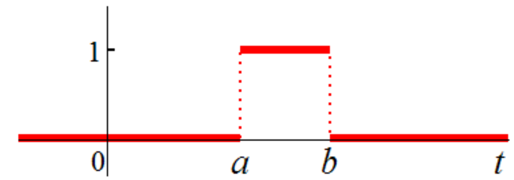
Unit Step Functions (of three types)



$$u_a(t) = u(t-a)$$
$$= \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



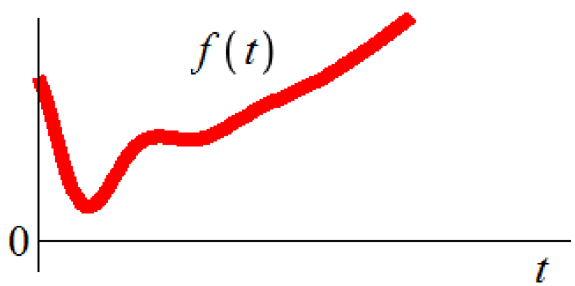
$$1 - u_a(t) = 1 - u(t-a)$$
$$= \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$$



$$u_a(t) - u_b(t)$$
$$= u(t-a) - u(t-b)$$
$$= \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

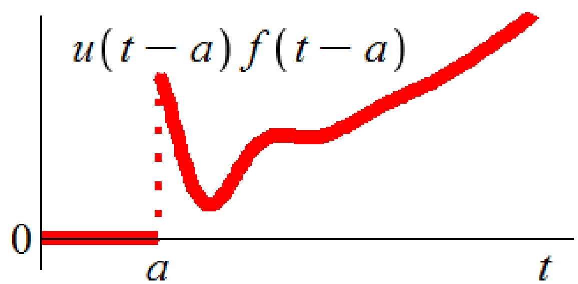
Laplace Transform Formula: Let $a > 0$.

If



$$\boxed{f(t)} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{F(s)}$$

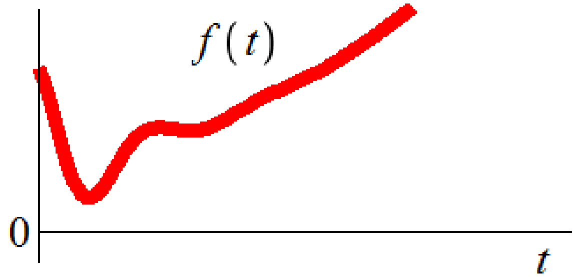
then



$$\boxed{\begin{aligned} u(t-a)f(t-a) &= u_a(t)f(t-a) \\ &= \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases} \end{aligned}} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{e^{-as}F(s)}$$

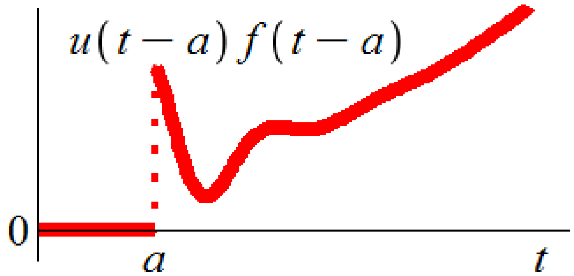
Laplace Transform Formula: Let $a > 0$.

If



$$\boxed{f(t)} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{F(s)}$$

then



$$\boxed{\begin{aligned} u(t-a)f(t-a) &= u_a(t)f(t-a) \\ &= \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases} \end{aligned}} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{e^{-as}F(s)}$$

Why?

By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$,

$$\mathcal{L}\{u(t-a)f(t-a)\} = \int_0^\infty e^{-st} u(t-a) f(t-a)\, dt$$

Substitution: $\tau = t - a$

$$= \int_0^a 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt = \int_a^\infty e^{-st} f(t-a)\, dt$$

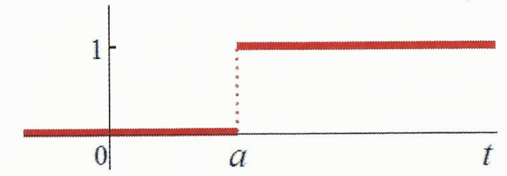
$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau$$

$$= \int_0^\infty e^{-as} e^{-s\tau} f(\tau)\, d\tau$$

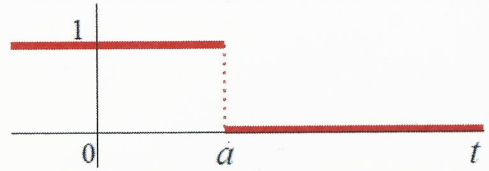
$$= e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-as} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-as} F(s).$$

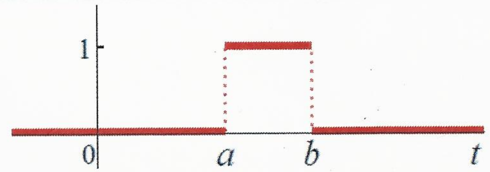
Replace τ by t



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$

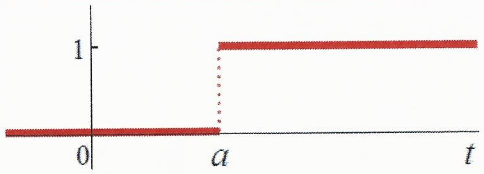


$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

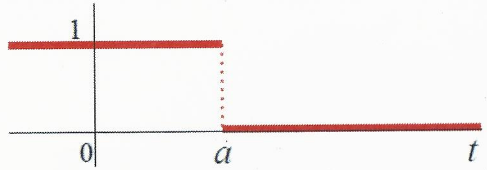
If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)\,f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$
$(a>0)$

$u(t-2)\,e^{7(t-2)} \xrightarrow{\mathcal{L}} e^{-2s}\dfrac{1}{s-7}$ used $e^{7t} \xrightarrow{\mathcal{L}} \dfrac{1}{s-7}$

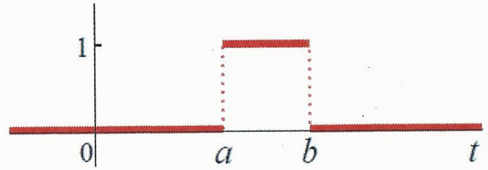
$u(t-3)\,(t-3)^4 \xrightarrow{\mathcal{L}} e^{-3s}\dfrac{24}{s^5}$ used $t^4 \xrightarrow{\mathcal{L}} \dfrac{4!}{s^5}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



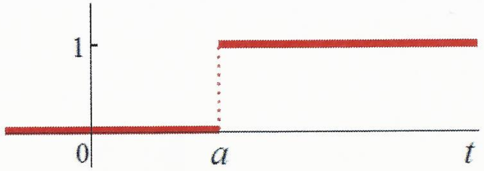
$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$

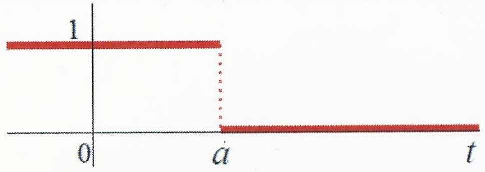
$(a>0)$

$$u(t-6)\left[(t-6)^2 - 4(t-6) + 7e^{-3(t-6)}\right] \xrightarrow{\mathcal{L}} e^{-6s}\left[\frac{2}{s^3} - \frac{4}{s^2} + \frac{7}{s+3}\right]$$

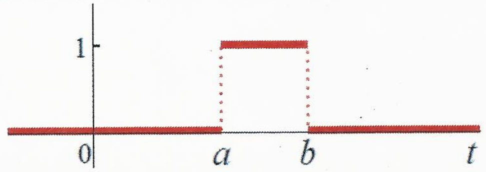
used $t^2 - 4t + 7e^{-3t} \xrightarrow{\mathcal{L}} \frac{2}{s^3} - \frac{4}{s^2} + \frac{7}{s+3}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

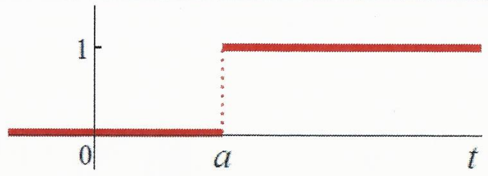
If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)\,f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$ $(a>0)$

$$u(t-3)\,t^2 = u(t-3)\left[(t-3)+3\right]^2 \xrightarrow{\mathcal{L}} e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$$

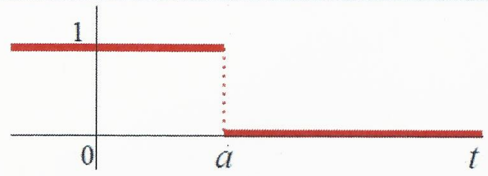
used $[t+3]^2 = t^2+6t+9 \xrightarrow{\mathcal{L}} \frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}$

$$u(t-7)\,e^{4t} = u(t-7)\,e^{4(t-7)+28} \xrightarrow{\mathcal{L}} e^{-7s}\,\frac{e^{28}}{s-4}$$

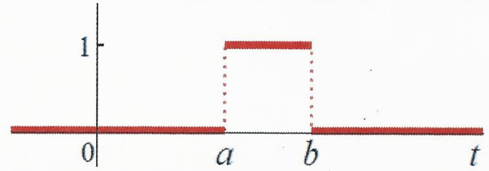
used $e^{4t+28} = e^{28}\,e^{4t} \xrightarrow{\mathcal{L}} \frac{e^{28}}{s-4}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)f(t-a) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} e^{-as}F(s)$ $(a>0)$

---

e.g. Find $\mathcal{L}^{-1}\left\{ e^{-2s}\frac{8}{s^6} - \frac{7e^{-3s}}{s^2+\pi^2} \right\}$.

Used
$\sin\pi(t-3) = \sin(\pi t - 3\pi)$
$= -\sin(\pi t)$

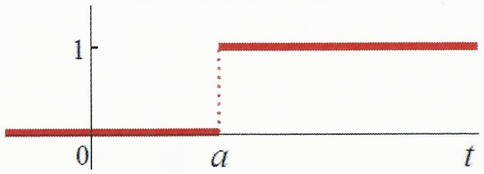
Solution

· $\mathcal{L}^{-1}\left\{\frac{8}{s^6}\right\} = \frac{8}{5!}t^5 = \frac{1}{15}t^5$

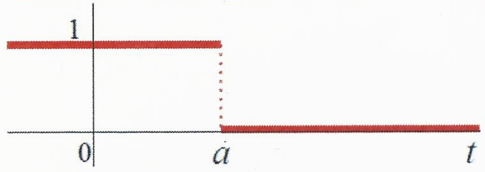
· $\mathcal{L}^{-1}\left\{\frac{7}{s^2+\pi^2}\right\} = \frac{7}{\pi}\sin(\pi t)$

$$\mathcal{L}^{-1}\left\{ e^{-2s}\frac{8}{s^6} - \frac{7e^{-3s}}{s^2+\pi^2} \right\} = u(t-2)\frac{1}{15}(t-2)^5 - u(t-3)\frac{7}{\pi}\sin\pi(t-3)$$

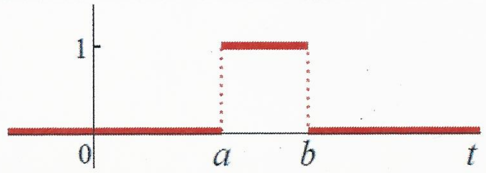
$$= u(t-2)\frac{1}{15}(t-2)^5 + u(t-3)\frac{7}{\pi}\sin(\pi t)$$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$$u_a(t) - u_b(t) = u(t-a) - u(t-b)$$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)\,f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$ $(a>0)$

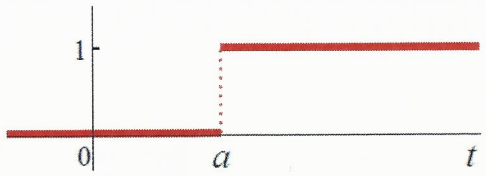
---

e.g.

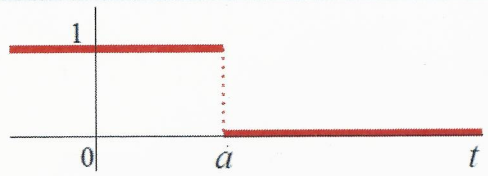
$$g(t) = \begin{cases} 0 & t<3, \\ e^{4(t-3)} & t \geq 3. \end{cases}$$ Find $\mathcal{L}\{g(t)\}$.

Solution : $g(t) = u(t-3)\,e^{4(t-3)} \xrightarrow{\mathcal{L}} e^{-3s}\dfrac{1}{s-4}$

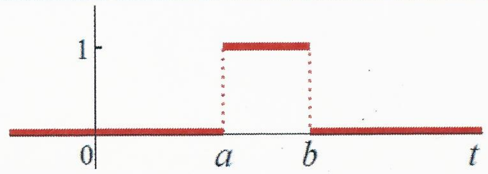
used $e^{4t} \xrightarrow{\mathcal{L}} \dfrac{1}{s-4}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

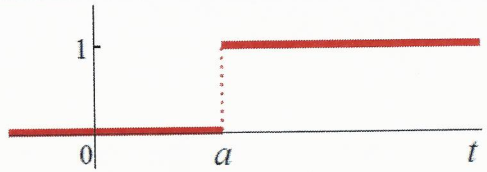
If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)\,f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$ $(a>0)$

e.g. Find the Laplace transform of $g(t) = \begin{cases} 0, & t<7 \\ (t-7)^2, & t \geq 7 \end{cases}$

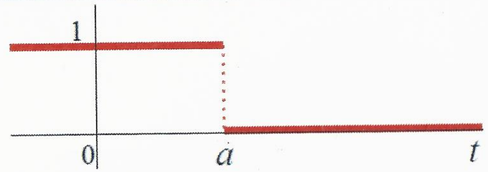
Solution:

$$g(t) = u(t-7)\,(t-7)^2 \xrightarrow{\mathcal{L}} e^{-7s}\frac{2}{s^3}$$

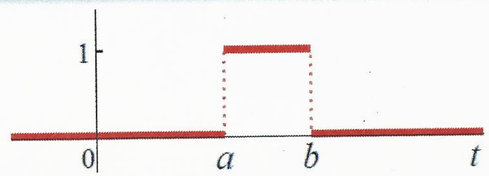
used $t^2 \xrightarrow{\mathcal{L}} \frac{2}{s^3}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

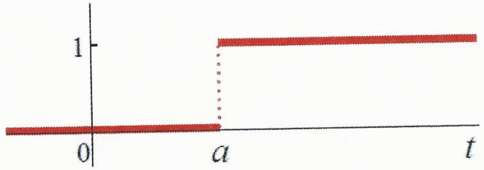
If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$
$(a>0)$

e.g. $g(t) = \begin{cases} 0 & t<3 \\ 2t+9 & t \geq 3 \end{cases}$. Find $\mathcal{L}\{g(t)\}$.

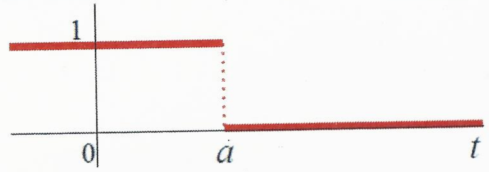
Solution: $g(t) = u(t-3)(2t+9) = u(t-3)[2(t-3)+15]$

$$\xrightarrow{\mathcal{L}} e^{-3s}\left[\frac{2}{s^2} + \frac{15}{s}\right]$$

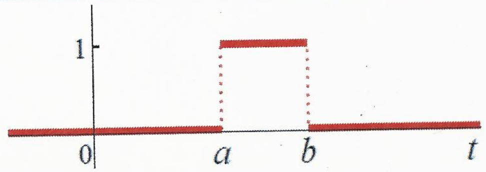
used $2t+15 \xrightarrow{\mathcal{L}} \frac{2}{s^2} + \frac{15}{s}$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$

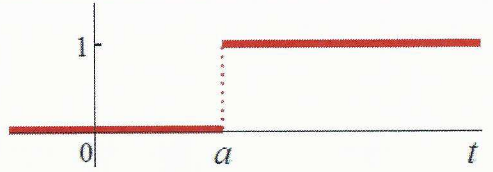


$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

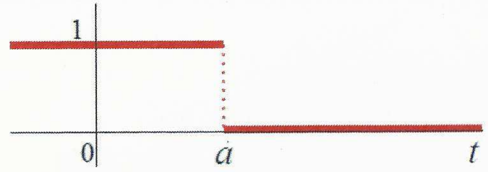
e.g. $g(t) = \begin{cases} 2t + e^{-3t} & t<4 \\ 0 & t \geq 4 \end{cases}$. Find $\mathcal{L}\{g(t)\}$.

<u>Solution</u>

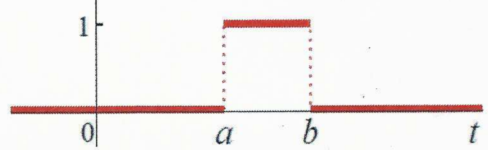
$$g(t) = [1 - u(t-4)](2t + e^{-3t})$$
$$= (2t + e^{-3t}) - u(t-4)(2t + e^{-3t})$$
$$= (2t + e^{-3t}) - u(t-4)[2(t-4) + 8 + e^{-3(t-4) - 12}]$$
$$\xrightarrow{\mathcal{L}} \frac{2}{s^2} + \frac{1}{s+3} - e^{-4s}\mathcal{L}\{2t + 8 + e^{-3t}e^{-12}\}$$
$$= \frac{2}{s^2} + \frac{1}{s+3} - e^{-4s}\left(\frac{2}{s^2} + \frac{8}{s} + \frac{e^{-12}}{s+3}\right)$$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $u(t-a)\,f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$ $(a>0)$

e.g. $g(t) = \begin{cases} t^2 & 3 \le t < 7 \\ 0 & \text{elsewhere} \end{cases}$. Find $\mathcal{L}\{g(t)\}$.

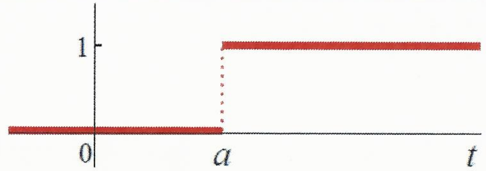
$\underline{\text{Solution}}$ $g(t) = [u(t-3) - u(t-7)]\,t^2 = u(t-3)t^2 - u(t-7)t^2$

$$= u(t-3)[(t-3)+3]^2 - u(t-7)[(t-7)+7]^2$$

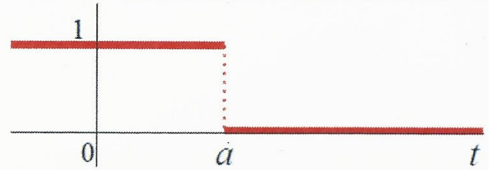
use $[t+3]^2 = t^2 + 6t + 9 \xrightarrow{\mathcal{L}} \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}$

use $[t+7]^2 = t^2 + 14t + 49 \xrightarrow{\mathcal{L}} \frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}$

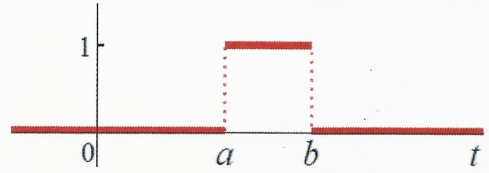
$$g(t) \xrightarrow{\mathcal{L}} e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right) - e^{-7s}\left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}\right)$$



$u_a(t) = u(t-a)$



$1 - u_a(t) = 1 - u(t-a)$



$u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$

e.g. $g(t) = \begin{cases} e^{2t} & t<3 \\ t & 3 \le t < 7 \\ 0 & 7 \le t < 9 \\ 4 & t \ge 9 \end{cases}$. Find $\mathcal{L}\{g(t)\}$.

Solution

$$g(t) = [1-u(t-3)]e^{2t} + [u(t-3)-u(t-7)]t + u(t-9)4$$

$$= e^{2t} + u(t-3)[-e^{2t}+t] - u(t-7)t + u(t-9)4$$

$$= e^{2t} + u(t-3)[-e^{2(t-3)}e^{6} + (t-3)+3] - u(t-7)[(t-7)+7] + u(t-9)4$$

$$\xrightarrow{\mathcal{L}} \frac{1}{s-2} + e^{-3s}\mathcal{L}\{-e^{2t}e^{6}+t+3\} - e^{-7s}\mathcal{L}\{t+7\} + e^{-9s}\mathcal{L}\{4\}$$

$$= \frac{1}{s-2} + e^{-3s}\left(-\frac{e^{6}}{s-2} + \frac{1}{s^2} + \frac{3}{s}\right) - e^{-7s}\left(\frac{1}{s^2}+\frac{7}{s}\right) + e^{-9s}\frac{4}{s}.$$