Laplace Transforms of Piecewise Defined Functions

The following formulas are in the text:

(i) If $a > 0$ and $g(t) = u(t-a)f(t-a) = \begin{cases} 0 & 0 \le t < a, \\ f(t-a) & t \ge a, \end{cases}$

then $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

(ii) If $a > 0$ and $\mathcal{L}^{-1}\{F(s)\} = f(t)$,

then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$.

The following may be helpful in practical computations.

(iii) If $a > 0$ and $g(t) = \begin{cases} 0 & 0 \le t < a, \\ f(t) & t \ge a, \end{cases}$

then $\mathcal{L}\{g(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$.

(iv) If $a > 0$ and $g(t) = \begin{cases} f(t) & 0 \le t < a, \\ 0 & t \ge a, \end{cases}$

then $\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t)\} - e^{-as}\mathcal{L}\{f(t+a)\}$.

(v) If $0 < a < b$ and $g(t) = \begin{cases} 0 & 0 \le t < a, \\ f(t) & a \le t < b, \\ 0 & t \ge b, \end{cases}$

then $\mathcal{L}\{g(t)\} = e^{-as}\mathcal{L}\{f(t+a)\} - e^{-bs}\mathcal{L}\{f(t+b)\}$.

(vi) If $0 < a_1 < a_2 < \cdots$ and $g(t) = \begin{cases} f_0(t) & 0 \le t < a_1, \\ f_1(t) & a_1 \le t < a_2, \\ f_2(t) & a_2 \le t < a_3, \\ \cdots, \end{cases}$

then

$$\begin{aligned} \mathcal{L}\{g(t)\} = {} & \mathcal{L}\{f_0(t)\} - e^{-a_1 s}\mathcal{L}\{f_0(t+a_1)\} \\ & + e^{-a_1 s}\mathcal{L}\{f_1(t+a_1)\} - e^{-a_2 s}\mathcal{L}\{f_1(t+a_2)\} \\ & + e^{-a_2 s}\mathcal{L}\{f_2(t+a_2)\} - e^{-a_3 s}\mathcal{L}\{f_2(t+a_3)\} \\ & + \cdots \end{aligned}$$

Example for (iii). For $g(t) = \begin{cases} 0 & 0 \le t < \pi/2, \\ \sin(7t) & t \ge \pi/2, \end{cases}$

we have

$$\begin{aligned}\mathcal{L}\{g(t)\} &= e^{-\pi s/2}\mathcal{L}\{\sin 7(t+\pi/2)\} = e^{-\pi s/2}\mathcal{L}\{\sin(7t + 7\pi/2)\} \\ &= e^{-\pi s/2}\mathcal{L}\{-\cos(7t)\} = -e^{-\pi s/2}\frac{s}{s^2+49}.\end{aligned}$$

Example for (iv). For $g(t) = \begin{cases} \sin(7t) & 0 \le t < \pi/2, \\ 0 & t \ge \pi/2, \end{cases}$

we have

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \mathcal{L}\{\sin(7t)\} - e^{-\pi s/2}\mathcal{L}\{\sin 7(t+\pi/2)\} = \mathcal{L}\{\sin(7t)\} - e^{-\pi s/2}\mathcal{L}\{-\cos(7t)\} \\ &= \frac{7}{s^2+49} + e^{-\pi s/2}\frac{s}{s^2+49}.\end{aligned}$$

Example for (v). For $g(t) = \begin{cases} 0 & 0 \le t < 5, \\ t^2 & 5 \le t < 8, \\ 0 & t \ge 8, \end{cases}$

we have

$$\begin{aligned}\mathcal{L}\{g(t)\} &= e^{-5s}\mathcal{L}\{(t+5)^2\} - e^{-8s}\mathcal{L}\{(t+8)^2\} \\ &= e^{-5s}\mathcal{L}\{t^2 + 10t + 25\} - e^{-8s}\mathcal{L}\{t^2 + 16t + 64\} \\ &= e^{-5s}\left(2s^{-3} + 10s^{-2} + 25s^{-1}\right) - e^{-8s}\left(2s^{-3} + 16s^{-2} + 64s^{-1}\right).\end{aligned}$$

Example for (vi). For $g(t) = \begin{cases} 7 & 0 \le t < 1, \\ t^2 & 1 \le t < 2, \\ 0 & 2 \le t < 3, \\ t^3 & 3 \le t < 4, \\ 0 & t \ge 4, \end{cases}$

we have

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \mathcal{L}\{7\} - e^{-s}\mathcal{L}\{7\} \\ &\quad + e^{-s}\mathcal{L}\{(t+1)^2\} - e^{-2s}\mathcal{L}\{(t+2)^2\} \\ &\quad + e^{-3s}\mathcal{L}\{(t+3)^3\} - e^{-4s}\mathcal{L}\{(t+4)^3\} \\ &= \mathcal{L}\{7\} - e^{-s}\mathcal{L}\{7\} + e^{-s}\mathcal{L}\{t^2 + 2t + 1\} - e^{-2s}\mathcal{L}\{t^2 + 4t + 4\} \\ &\quad + e^{-3s}\mathcal{L}\{t^3 + 9t^2 + 27t + 27\} - e^{-4s}\mathcal{L}\{t^3 + 12t^2 + 48t + 64\} \\ &= 7s^{-1} - e^{-s}\left(7s^{-1}\right) + e^{-s}\left(2s^{-3} + 2s^{-2} + s^{-1}\right) - e^{-2s}(2s^{-3} + 4s^{-2} + 4s^{-1}) \\ &\quad + e^{-3s}(6s^{-4} + 18s^{-3} + 27s^{-2} + 27s^{-1}) - e^{-4s}(6s^{-4} + 24s^{-3} + 48s^{-2} + 64s^{-1}) \\ &= 7s^{-1} + e^{-s}(2s^{-3} + 2s^{-2} - 6s^{-1}) - e^{-2s}(2s^{-3} + 4s^{-2} + 4s^{-1}) \\ &\quad + e^{-3s}(6s^{-4} + 18s^{-3} + 27s^{-2} + 27s^{-1}) - e^{-4s}(6s^{-4} + 24s^{-3} + 48s^{-2} + 64s^{-1})\end{aligned}$$