Homogeneous Linear Systems of Differential Equations with Constant Coefficients

Objective: Solve

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,\\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,\\
&\vdots\\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.
\end{aligned}$$

This is a system of n differential equations for n unknowns $x_1(t), \cdots, x_n(t)$. We can put the system in an equivalent matrix form:

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}},$$

with unknown being the vector function $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

Solution Method: Suppose that A is diagonalizable; that is, there are an invertible matrix P and a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ such that $A = PDP^{-1}$. In this case, set

$$\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t).$$

The system for $\vec{\mathbf{u}}(t)$ becomes

$$\frac{d\vec{\mathbf{u}}}{dt} = P^{-1}AP\vec{\mathbf{u}} = D\vec{\mathbf{u}}, \qquad \text{or, equivalently, } \begin{cases} \dfrac{du_1}{dt} = \lambda_1 u_1, \\ \quad\vdots \\ \dfrac{du_n}{dt} = \lambda_n u_n. \end{cases}$$

The last system is a completely decoupled system, which we can easily solve to get solutions

$$u_1(t) = C_1 e^{\lambda_1 t}, \cdots, u_n(t) = C_n e^{\lambda_n t}.$$

Going back to $\vec{\mathbf{x}}$ we get the solutions to the given system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$:

$$\vec{\mathbf{x}}(t) = P \begin{bmatrix} C_1 e^{\lambda_1 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{bmatrix}.$$

An equivalent but even simpler formulation of the solution method: First construct a basis of $\mathbb{R}^n$ consisting of eigenvectors of A:

$$\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_n,$$

corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then the solutions of the system of differenttial equations are:

$$\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + \cdots + C_n e^{\lambda_n t}\vec{\mathbf{v}}_n,$$

where $C_1, C_2, \cdots, C_n$ are free parameters.

What if not diagonalizable? In the case A is not diagonalizable (i.e., you cannot find n linearly independent eigenvectors of A), the system cannot be completely decoupled as above and the problem is a little harder. But even in such cases, the eigenvectors provide a partial but crucial help. In general, if λ is an eigenvalue of A with multiplicity k, we can obtain k linearly independent solutions by considering solutions of the following form:

$$\vec{\mathbf{x}}(t) = e^{\lambda t}\vec{\mathbf{v}}_1 + te^{\lambda t}\vec{\mathbf{v}}_2 + \cdots + t^{k-1}e^{\lambda t}\vec{\mathbf{v}}_k.$$

Plugging this into the system of differential equations and comparing the coefficients, we will get a system of linear equations for vectors $\vec{\mathbf{v}}_1, \cdots, \vec{\mathbf{v}}_k$. The solution space of this linear system turns out to be a k dimensional subspace and thus yields a k-parameter family of solutions of the differential equations. See Examples 2 and 3.

EXAMPLE 1 (Simple Eigenvalues). Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$ for $A = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix}$.

- The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$.
- For each eigenvalue we can find a corresponding eigenvector:

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Here, $\vec{\mathbf{v}}_j$ is an eigenvector associated to eigenvalue λ_j ($j = 1, 2, 3$).

- This shows matrix A is diagonalizable: $A = PDP^{-1}$ with

$$P = [\,\vec{\mathbf{v}}_1 \quad \vec{\mathbf{v}}_2 \quad \vec{\mathbf{v}}_3\,] = \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

- Now set $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

$$\frac{d\vec{\mathbf{u}}}{dt} = D\vec{\mathbf{u}}, \qquad \text{or, equivalently,} \begin{cases} du_1/dt = u_1, \\ du_2/dt = 2u_2, \\ du_3/dt = -u_3. \end{cases}$$

Solving this decoupled system, we obtain

$$u_1(t) = C_1 e^t, u_2(t) = C_2 e^{2t}, u_3(t) = C_3 e^{-t}.$$

Finally, the solutions to the given system are

$$\vec{\mathbf{x}}(t) = P \begin{bmatrix} C_1 e^t \\ C_2 e^{2t} \\ C_3 e^{-t} \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix},$$

where C_1, C_2, C_3 are free parameters.

EXAMPLE 2 (Repeated Eigenvalues, Diagonalizable). Solve the initial value probelm

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \qquad \text{for } A = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix}.$$

- The eigenvalues of A are $\lambda_1 = -1, \lambda_2 = \lambda_3 = 2$.
- Set solutions in the following form:

$$\vec{\mathbf{x}}(t) = e^{-t}\vec{\mathbf{a}} + e^{2t}\vec{\mathbf{b}} + te^{2t}\vec{\mathbf{c}} = e^{-t}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + e^{2t}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + te^{2t}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Plug this into $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$:

$$-e^{-t}\vec{\mathbf{a}} + 2e^{2t}\vec{\mathbf{b}} + e^{2t}\vec{\mathbf{c}} + 2te^{2t}\vec{\mathbf{c}} = e^{-t}A\vec{\mathbf{a}} + e^{2t}A\vec{\mathbf{b}} + te^{2t}A\vec{\mathbf{c}}.$$

A comparison of coefficients gives:

$$\begin{cases} A\vec{\mathbf{a}} = -\vec{\mathbf{a}} \\ A\vec{\mathbf{b}} = 2\vec{\mathbf{b}} + \vec{\mathbf{c}} \\ A\vec{\mathbf{c}} = 2\vec{\mathbf{c}} \end{cases} \Leftrightarrow \begin{cases} (A+I)\vec{\mathbf{a}} = 0 \\ \begin{bmatrix} A-2I & -I \\ 0 & A-2I \end{bmatrix}\begin{bmatrix} \vec{\mathbf{b}} \\ \vec{\mathbf{c}} \end{bmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix} -6 & -9 & 9 \\ 3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \\ \begin{bmatrix} -9 & -9 & 9 & -1 & 0 & 0 \\ 3 & 3 & -3 & 0 & -1 & 0 \\ -3 & -3 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -9 & -9 & 9 \\ 0 & 0 & 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & -3 & -3 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \end{cases}$$

Solve this system of linear equations for vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$:

$$\vec{\mathbf{a}} = a_3\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{b}} = b_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{\mathbf{c}} = 0$$

where a_3, b_2, b_3 are arbitrary constants.

- The general solutions to the given system are

$$\vec{\mathbf{x}}(t) = a_3e^{-t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + b_2e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b_3e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad (*)$$

where a_3, b_2, b_3 are free parameters.

- Finally, let's get the solution to the initial value problem. Let $t = 0$ in equation $(*)$:

$$a_3\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + b_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} a_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 7 \end{bmatrix}.$$

Plugging this back in $(*)$, we obtain the answer

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} -9e^{-t} + 12e^{2t} \\ 3e^{-t} - 5e^{2t} \\ -3e^{-t} + 7e^{2t} \end{bmatrix}.$$

EXAMPLE 3 (Repeated Eigenvalues, Not Diagonalizable).

Solve $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$ for $A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix}$.

- The eigenvalues of A are $\lambda_1 = -1, \lambda_2 = \lambda_3 = -3$.
- Set solutions in the following form:

$$\vec{\mathbf{x}}(t) = e^{-t}\vec{\mathbf{a}} + e^{-3t}\vec{\mathbf{b}} + te^{-3t}\vec{\mathbf{c}} = e^{-t} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + e^{-3t} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + te^{-3t} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Plug this into $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$:

$$-e^{-t}\vec{\mathbf{a}} - 3e^{-3t}\vec{\mathbf{b}} + e^{-3t}\vec{\mathbf{c}} - 3te^{-3t}\vec{\mathbf{c}} = e^{-t}A\vec{\mathbf{a}} + e^{-3t}A\vec{\mathbf{b}} + te^{-3t}A\vec{\mathbf{c}}.$$

A comparison of coefficients gives:

$$\begin{cases} A\vec{\mathbf{a}} = -\vec{\mathbf{a}} \\ A\vec{\mathbf{b}} = -3\vec{\mathbf{b}} + \vec{\mathbf{c}} \\ A\vec{\mathbf{c}} = -3\vec{\mathbf{c}} \end{cases} \Leftrightarrow \begin{cases} (A+I)\vec{\mathbf{a}} = 0 \\ \begin{bmatrix} A+3I & -I \\ 0 & A+3I \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}} \\ \vec{\mathbf{c}} \end{bmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix} -4 & -8 & 4 \\ 2 & 4 & -2 \\ 6 & 14 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \\ \begin{bmatrix} -2 & -8 & 4 & -1 & 0 & 0 \\ 2 & 6 & -2 & 0 & -1 & 0 \\ 6 & 14 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & -8 & 4 \\ 0 & 0 & 0 & 2 & 6 & -2 \\ 0 & 0 & 0 & 6 & 14 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \end{cases}$$

Solve this system of linear equations for vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$:

$$\vec{\mathbf{a}} = a_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{b}} = b_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{c}} = c_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

where a_3, b_3, c_3 are arbitrary constants.

- The general solutions of the system of differential equations are

$$\begin{aligned}\vec{\mathbf{x}}(t) &= e^{-t}a_3\begin{bmatrix}3\\-1\\1\end{bmatrix} + e^{-3t}\left(b_3\begin{bmatrix}-2\\1\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1/2\\0\end{bmatrix}\right) + te^{-3t}c_3\begin{bmatrix}-2\\1\\1\end{bmatrix}\\ &= a_3e^{-t}\begin{bmatrix}3\\-1\\1\end{bmatrix} + b_3e^{-3t}\begin{bmatrix}-2\\1\\1\end{bmatrix} + c_3e^{-3t}\begin{bmatrix}-1-2t\\1/2+t\\t\end{bmatrix},\end{aligned}$$

where a_3, b_3, c_3 are arbitrary constants.

EXAMPLE 4 (Complex Eigenvalues). Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$ for $A = \begin{bmatrix}3 & 22 & -36\\2 & 5 & -18\\2 & 7 & -17\end{bmatrix}$.

- The eigenvalues of A are $\lambda_1 = -2+3i, \lambda_2 = -2-3i, \lambda_3 = -5$.
- For each eigenvalue we can find a corresponding eigenvector:

$$\vec{\mathbf{v}}_1 = \begin{bmatrix}4-2i\\1+i\\1\end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix}4+2i\\1-i\\1\end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix}-1\\2\\1\end{bmatrix}.$$

Here, $\vec{\mathbf{v}}_j$ is an eigenvector for eigenvalue λ_j $(j = 1,2,3)$, obtained by solving $(A - \lambda_j I)\vec{\mathbf{v}} = 0$.

The solutions to the given system are

$$\vec{\mathbf{x}}(t) = C_1e^{\lambda_1 t}\vec{\mathbf{v}}_1 + C_2e^{\lambda_2 t}\vec{\mathbf{v}}_2 + C_3e^{\lambda_3 t}\vec{\mathbf{v}}_3 = C_1e^{(-2+3i)t}\begin{bmatrix}4-2i\\1+i\\1\end{bmatrix} + C_2e^{(-2-3i)t}\begin{bmatrix}4+2i\\1-i\\1\end{bmatrix} + C_3e^{-5t}\begin{bmatrix}-1\\2\\1\end{bmatrix},$$

where C_1, C_2, C_3 are free parameters.

Via Euler's formula, we can write down an alternative expression of general solutions:

$$\begin{aligned}\vec{\mathbf{x}}(t) &= C_1\,\mathrm{Re}\left\{e^{\lambda_1 t}\vec{\mathbf{v}}_1\right\} + C_2\,\mathrm{Im}\left\{e^{\lambda_1 t}\vec{\mathbf{v}}_1\right\} + C_3e^{\lambda_3 t}\vec{\mathbf{v}}_3\\ &= C_1e^{-2t}\left(\cos(3t)\begin{bmatrix}4\\1\\1\end{bmatrix} - \sin(3t)\begin{bmatrix}-2\\1\\0\end{bmatrix}\right) + C_2e^{-2t}\left(\sin(3t)\begin{bmatrix}4\\1\\1\end{bmatrix} + \cos(3t)\begin{bmatrix}-2\\1\\0\end{bmatrix}\right) + C_3e^{-5t}\begin{bmatrix}-1\\2\\1\end{bmatrix},\end{aligned}$$

where C_1, C_2, C_3 are free parameters.

EXERCISES

[1] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

[2] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -4 & 12 \\ -3 & 8 \end{bmatrix} \vec{\mathbf{x}}$.

[3] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

[4] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 5 & 4 & -2 \\ -12 & -9 & 4 \\ -12 & -8 & 3 \end{bmatrix} \vec{\mathbf{x}}$.

[5] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 9 & 7 & -3 \\ -16 & -12 & 5 \\ -8 & -5 & 2 \end{bmatrix} \vec{\mathbf{x}}$.

[6] Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 5 & -2 & -10 \\ 4 & 1 & 0 \\ 4 & 0 & -7 \end{bmatrix} \vec{\mathbf{x}}$.

Answers:

[1] $\vec{\mathbf{x}}(t) = \begin{bmatrix} 9e^{4t} - 4e^{2t} \\ 3e^{4t} - 4e^{2t} \end{bmatrix}$

[2] $\vec{\mathbf{x}}(t) = C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1/3 + 2t \\ t \end{bmatrix}$ where C_1, C_2 are arbitrary constants

[3] $\vec{\mathbf{x}}(t) = e^{t} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + e^{2t} \begin{bmatrix} -4 \\ -6 \\ -6 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

[4] $\vec{\mathbf{x}}(t) = C_1 e^{t} \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$ where C_1, C_2, C_3 are arbitrary constants

[5] $\vec{\mathbf{x}}(t) = C_1 e^{t} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} -2 + t \\ 3 - t \\ t \end{bmatrix}$ where C_1, C_2, C_3 are arbitrary constants

[6] $\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{(1+4i)t} \begin{bmatrix} 2 + i \\ 1 - 2i \\ 1 \end{bmatrix} + C_3 e^{(1-4i)t} \begin{bmatrix} 2 - i \\ 1 + 2i \\ 1 \end{bmatrix}$

where C_1, C_2, C_3 are arbitrary constants, or equivalently,

$$\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{t} \left(\cos(4t) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right) + C_3 e^{t} \left(\sin(4t) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right)$$