The Green's Function

1. Find the Green's function for $y'' - 4y = f(x), \quad y'(0) = 0, y(2) = 0.$

2. Find the Green's function for $y'' = f(x), \quad y(-3) = 0, y'(1) + y(1) = 0.$

3. (a) Verify that $y_1(x) = x$ is a solution of

$$x^2y'' + xy' - y = 0. \tag{$*$}$$

   (b) Use $y_1(x)$ in (a) to find the general solutions of $(*)$.

   (c) Find the Green's function for

$$x^2y'' + xy' - y = f(x), \quad y(1) = 0, y(2) = 0.$$

   (d) Use the Green's function obtained in (c) to solve

$$x^2y'' + xy' - y = x^2e^{-x}, \quad y(1) = 0, y(2) = 0.$$

See next page for the answers

Answers:

1.

$$G(x,s)=\begin{cases}\dfrac{(e^{2s}+e^{-2s})(e^{2x}-e^{8-2x})}{4(e^8+1)} & \text{for } 0\le s\le x\le 2,\\ \dfrac{(e^{2x}+e^{-2x})(e^{2s}-e^{8-2s})}{4(e^8+1)} & \text{for } 0\le x\le s\le 2.\end{cases}$$

2.

$$G(x,s)=\begin{cases}\dfrac{(s+3)(x-2)}{5} & \text{for } -3\le s\le x\le 1,\\ \dfrac{(x+3)(s-2)}{5} & \text{for } -3\le x\le s\le 1.\end{cases}$$

3. (a) Just plug y_1 in equation $(*)$. I'll skip the details.

   (b) $y(x)=C_1x+C_2x^{-1}$

   (c)

$$G(x,s)=\begin{cases}\dfrac{(s^2-1)(x-4x^{-1})}{6s^2} & \text{for } 1\le s\le x\le 2,\\ \dfrac{(x-x^{-1})(s^2-4)}{6s^2} & \text{for } 1\le x\le s\le 2.\end{cases}$$

   (d) $y(x)=e^{-x}+x^{-1}e^{-x}+e^{-2}x^{-1}-e^{-2}x+\dfrac{2}{3}e^{-1}x-\dfrac{8}{3}e^{-1}x^{-1}$