$\boxed{\text{Evaluate } e^{tA}, \text{ using the Laplace transform}}$

$$e^{tA} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}.$$

Why?

- Consider the initial value problem (✱) $\frac{d\mathbf{x}}{dt}(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$
- On one hand, $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, by definition of e^{tA}.
- On the other hand, we can solve (✱) using $\mathcal{L}$:

$$(*) \xrightarrow{\mathcal{L}} s\mathbf{X}(s) - \mathbf{x}_0 = A\mathbf{X}(s)$$

$\Downarrow$ Solve

$$(sI-A)\mathbf{X}(s) = \mathbf{x}_0,$$

$$\mathbf{X}(s) = (sI-A)^{-1}\mathbf{x}_0 \xrightarrow{\mathcal{L}^{-1}} \mathbf{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}\mathbf{x}_0.$$

Example Let $A=\begin{bmatrix}-3 & 2\\ 1 & -4\end{bmatrix}$. Evaluate e^{tA}, using the Laplace transform.

Solution

- $(sI-A)^{-1}=\begin{bmatrix}s+3 & -2\\ -1 & s+4\end{bmatrix}^{-1}=\dfrac{1}{s^2+7s+10}\begin{bmatrix}s+4 & 2\\ 1 & s+3\end{bmatrix}=\dfrac{1}{(s+2)(s+5)}\begin{bmatrix}s+4 & 2\\ 1 & s+3\end{bmatrix}$.

- Partial Fraction: Set $\dfrac{1}{(s+2)(s+5)}\begin{bmatrix}s+4 & 2\\ 1 & s+3\end{bmatrix}=\dfrac{1}{s+2}B_1+\dfrac{1}{s+5}B_2$.

  Multiply both sides by $(s+2)(s+5)$: $(*)$ $\begin{bmatrix}s+4 & 2\\ 1 & s+3\end{bmatrix}=(s+5)B_1+(s+2)B_2$

  Let $s=-2$ in $(*)$: $\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}=3B_1 \Rightarrow B_1=\begin{bmatrix}2/3 & 2/3\\ 1/3 & 1/3\end{bmatrix}$.

  Let $s=-5$ in $(*)$: $\begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}=-3B_2 \Rightarrow B_2=\begin{bmatrix}1/3 & -2/3\\ -1/3 & 2/3\end{bmatrix}$.

  Hence, $(sI-A)^{-1}=\dfrac{1}{s+2}\begin{bmatrix}2/3 & 2/3\\ 1/3 & 1/3\end{bmatrix}+\dfrac{1}{s+5}\begin{bmatrix}1/3 & -2/3\\ -1/3 & 2/3\end{bmatrix}$.

- $e^{tA}=\mathcal{L}^{-1}\{(sI-A)^{-1}\}=e^{-2t}\begin{bmatrix}2/3 & 2/3\\ 1/3 & 1/3\end{bmatrix}+e^{-5t}\begin{bmatrix}1/3 & -2/3\\ -1/3 & 2/3\end{bmatrix}$.