$\boxed{\text{Matrix Exponential}}$

What is e^{tA}?

where A is an $n \times n$ matrix.

- Express the solution of $\begin{cases} \dfrac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$ in the form $\vec{x}(t) = \boxed{\quad}\,\vec{x}_0$

This is e^{tA}.

i.e. $\vec{x}(t) = e^{tA}\vec{x}_0$ is the sol. of the init. value prob.

- Suppose that the gen. sol's of $\dfrac{d\vec{x}}{dt} = A\vec{x}$ are:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t).$$

$\Rightarrow$ $M(t) = [\vec{x}_1(t) \;\; \vec{x}_2(t) \;\; \cdots\cdots \;\; \vec{x}_n(t)]$ is a fundamental matrix.

$$\Rightarrow e^{tA} = M(t)\,M(0)^{-1}$$

- $$e^{tA} = \sum_{j=0}^{\infty} \frac{t^j}{j!}A^j = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \cdots\cdots$$

Properties of matrix exponential :

① $e^{t(A+B)} \neq e^{tA} e^{tB}$ in general.

② $e^{t(A+B)} = e^{tA} e^{tB}$ if and only if $AB = BA$

③ $e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}$

$$= e^{\lambda t} \left\{ I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \frac{t^3}{3!}(A-\lambda I)^3 + \cdots\cdots \right\}$$

④ If $(A-\lambda I)^k \vec{v} = 0$,

then $e^{tA}\vec{v} = e^{\lambda t}\left\{ I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1} \right\} \vec{v}$

⑤ If $A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$, then $e^{tA} = \begin{bmatrix} e^{a_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{a_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{a_n t} \end{bmatrix}$

(diagonal)

Example 1 (e^{tA} with A having distinct real eigenvalues)

Find e^{tA} for $A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$.

Solution: Key Formula: $e^{tA} = M(t)M(0)^{-1}$, where $M(t)$ is a fundamental matrix.

· Eigenvalues of A: $\det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & 2 \\ 1 & -4-\lambda \end{bmatrix} = \lambda^2+7\lambda+10 = (\lambda+2)(\lambda+5)$

$\lambda_1 = -2, \quad \lambda_2 = -5$.

· Eigenvectors for $\lambda_1 = -2$

$(A+2I)\vec{x} = \vec{0} \iff \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 = 2x_2 \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

· Eigenvectors for $\lambda_2 = -5$

$(A+5I)\vec{x} = \vec{0} \iff \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 = -x_2 \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

· Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x}$: $\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

· A Fundamental Matrix $M(t) = \begin{bmatrix} 2e^{-2t} & -e^{-5t} \\ e^{-2t} & e^{-5t} \end{bmatrix}$

· $e^{tA} = M(t)M(0)^{-1} = \begin{bmatrix} 2e^{-2t} & -e^{-5t} \\ e^{-2t} & e^{-5t} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2e^{-2t} & -e^{-5t} \\ e^{-2t} & e^{-5t} \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{3}e^{-2t}+\frac{1}{3}e^{-5t} & \frac{2}{3}e^{-2t}-\frac{2}{3}e^{-5t} \\ \frac{1}{3}e^{-2t}-\frac{1}{3}e^{-5t} & \frac{1}{3}e^{-2t}+\frac{2}{3}e^{-5t} \end{bmatrix}$.

Example 2 (e^{tA} with A having complex eigenvalues)

Evaluate e^{tA} for $A = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix}$.

$\boxed{\text{Solution}}$ Eigenvalues of A: $\det(A-\lambda I) = \det\begin{bmatrix} -5-\lambda & -39 \\ 6 & 1-\lambda \end{bmatrix} = \lambda^2+4\lambda+229 \Rightarrow \lambda_{1,2} = -2 \pm 15i$

- Eigenvectors for $\lambda_1 = -2+15i$ $\quad [A-(-2+15i)I]\vec{x} = \vec{0}$

$$\begin{bmatrix} -3-15i & -39 \\ 6 & 3-15i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow 6x_1 + (3-15i)x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -\frac{1}{2}+\frac{5}{2}i \\ 1 \end{bmatrix}$$

- Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x}$: $\vec{x}(t) = C_1 e^{-2t}\left\{\cos 15t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin 15t\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right\} + C_2 e^{-2t}\left\{\sin 15t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos 15t\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right\}$

- A Fundamental Matrix $M(t) = \begin{bmatrix} -\frac{1}{2}e^{-2t}\cos 15t - \frac{5}{2}e^{-2t}\sin 15t & -\frac{1}{2}e^{-2t}\sin 15t + \frac{5}{2}e^{-2t}\cos 15t \\ e^{-2t}\cos 15t & e^{-2t}\sin 15t \end{bmatrix}$.

$$M(0)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{5}{2} \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

- Matrix Exponential $e^{tA} = M(t)M(0)^{-1}$

$$= \begin{bmatrix} -\frac{1}{2}e^{-2t}\cos 15t - \frac{5}{2}e^{-2t}\sin 15t & -\frac{1}{2}e^{-2t}\sin 15t + \frac{5}{2}e^{-2t}\cos 15t \\ e^{-2t}\cos 15t & e^{-2t}\cos 15t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}\cos 15t - \frac{1}{5}e^{-2t}\sin 15t & -\frac{13}{5}e^{-2t}\sin 15t \\ \frac{2}{5}e^{-2t}\sin 15t & e^{-2t}\cos 15t + \frac{1}{5}e^{-2t}\sin 15t \end{bmatrix}$$

Example 2 Evaluate e^{tA} for $A = \begin{bmatrix} -5 & -34 \\ 6 & 1 \end{bmatrix}$.

$\boxed{\text{Alternative Solution}}$ Eigenvalues of A: $\det(A-\lambda I) = \cdots = \lambda^2 + 4\lambda + 229 \Rightarrow \lambda_{1,2} = -2 \pm 15i$

- Eigenvectors for $\lambda_1 = -2+15i$

$[A-(-2+15i)I]\vec{x} = \vec{0} \Rightarrow \cdots$ Compute $\cdots \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2}+\frac{5}{2}i \\ 1 \end{bmatrix}$

- Eigenvectors for $\lambda_2 = -2-15i$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2}-\frac{5}{2}i \\ 1 \end{bmatrix}$

- A Fundamental Matrix:

$$M(t) = \begin{bmatrix} \left(-\frac{1}{2}+\frac{5}{2}i\right)e^{(-2+15i)t} & \left(-\frac{1}{2}-\frac{5}{2}i\right)e^{(-2-15i)t} \\ e^{(-2+15i)t} & e^{(-2-15i)t} \end{bmatrix}, \quad M(0)^{-1} = \begin{bmatrix} -\frac{1}{2}+\frac{5}{2}i & -\frac{1}{2}-\frac{5}{2}i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5}i & \frac{1}{2}-\frac{1}{10}i \\ \frac{1}{5}i & \frac{1}{2}+\frac{1}{10}i \end{bmatrix}$$

- Matrix Exponential (expressed using complex exp. functions)

$$e^{tA} = M(t)\,M(0)^{-1} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{5}{2}i\right)e^{(-2+15i)t} & \left(-\frac{1}{2}-\frac{5}{2}i\right)e^{(-2-15i)t} \\ e^{(-2+15i)t} & e^{(-2-15i)t} \end{bmatrix} \begin{bmatrix} -\frac{1}{5}i & \frac{1}{2}-\frac{1}{10}i \\ \frac{1}{5}i & \frac{1}{2}+\frac{1}{10}i \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}+\frac{1}{10}i\right)e^{(-2+15i)t} + \left(\frac{1}{2}-\frac{1}{10}i\right)e^{(-2-15i)t} & \frac{13}{10}i\,e^{(-2+15i)t} - \frac{13}{10}i\,e^{(-2-15i)t} \\ -\frac{1}{5}i\,e^{(-2+15i)t} + \frac{1}{5}i\,e^{(-2-15i)t} & \left(\frac{1}{2}-\frac{1}{10}i\right)e^{(-2+15i)t} + \left(\frac{1}{2}+\frac{1}{10}i\right)e^{(-2-15i)t} \end{bmatrix}$$

Example 3 (e^{tA} with A having distinct eigenvalues)

(a) Evaluate e^{tA} for $A = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix}$ (b) Solve $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

$\boxed{\text{Solution to (a)}}$ Compute eigenvalues & eigenvectors:

$\lambda_1 = 1$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. $\lambda_3 = -1$, $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

- Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x}$ $\quad \vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

- A Fundamental Matrix

$$M(t) = \begin{bmatrix} e^t & -e^{2t} & \frac{1}{2}e^{-t} \\ -e^t & 2e^{2t} & -\frac{1}{2}e^{-t} \\ e^t & e^{2t} & e^{-t} \end{bmatrix}, \qquad M(0)^{-1} = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -\frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 0 \\ -6 & -4 & 2 \end{bmatrix}$$

- Matrix Exp

$$e^{tA} = M(t)\,M(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} & \frac{1}{2}e^{-t} \\ -e^t & 2e^{2t} & -\frac{1}{2}e^{-t} \\ e^t & e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 0 \\ -6 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5e^t - e^{2t} - 3e^{-t} & 3e^t - e^{2t} - 2e^{-t} & -e^t + e^{-t} \\ -5e^t + 2e^{2t} + 3e^{-t} & -3e^t + 2e^{2t} + 2e^{-t} & e^t - e^{-t} \\ 5e^t + e^{2t} - 6e^{-t} & 3e^t + e^{2t} - 4e^{-t} & -e^t + 2e^{-t} \end{bmatrix}$$

$\boxed{\text{Solution to (b)}}$ $\quad \vec{x}(t) = e^{tA} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \cdots\cdots \underset{\text{multiply}}{\text{Compute}} \cdots\cdots = \begin{bmatrix} -11e^t + e^{2t} + 8e^{-t} \\ 11e^t - 2e^{2t} - 8e^{-t} \\ -11e^t - e^{2t} + 16e^{-t} \end{bmatrix}$.

Example 4 (e^{tA} with A having a repeated eigenvalue)

repeated multiplicity 2

Evalute e^{tA} for $A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix}$.

Solution

Compute Eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = \lambda_3 = -3$

- Eigenvectors for $\lambda_1 = -1$: $(A+I)\vec{x} = \vec{0}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.
- Generalized Eigenvectors for $\lambda_2 = \lambda_3 = -3$ $\quad (A+3I)^2 \vec{x} = \vec{0}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- Gen. Sol's of $\vec{x}' = A\vec{x}$

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \{ I + t(A+3I) \} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \{ I + t(A+3I) \} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= c_1 e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2-4t \\ 1+2t \\ 2t \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 4t \\ -2t \\ 1-2t \end{bmatrix}$$

- A Fundamental Matrix $M(t) = \begin{bmatrix} 3e^{-t} & (-2-4t)e^{-3t} & 4te^{-3t} \\ -e^{-t} & (1+2t)e^{-3t} & -2te^{-3t} \\ e^{-t} & 2te^{-3t} & (1-2t)e^{-3t} \end{bmatrix}$, $M(0)^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

- Matrix Exp

$$e^{tA} = M(t)M(0)^{-1} = \begin{bmatrix} 3e^{-t} & (-2-4t)e^{-3t} & 4te^{-3t} \\ -e^{-t} & (1+2t)e^{-3t} & -2te^{-3t} \\ e^{-t} & 2te^{-3t} & (1-2t)e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} + (-2-8t)e^{-3t} & 6e^{-t} + (-6-20t)e^{-3t} & 4te^{-3t} \\ -e^{-t} + (1+4t)e^{-3t} & -2e^{-t} + (3+10t)e^{-3t} & -2te^{-3t} \\ e^{-t} + (-1+4t)e^{-3t} & 2e^{-t} + (-2+10t)e^{-3t} & (1-2t)e^{-3t} \end{bmatrix}$$

Example 5 (e^{tA} with all the eigenvalues equal)

Evaluate e^{tA}, for $A = \begin{bmatrix} 11/2 & 1 & -1/2 \\ -1 & 7 & 1 \\ 1/2 & -1 & 17/2 \end{bmatrix}$.

$\boxed{\text{Solution}}$ Eigenvalues $\det(A-\lambda I) = \cdots = -\lambda^3+21\lambda^2-147\lambda+343 = -(\lambda-7)^3$.

$\lambda_1=\lambda_2=\lambda_3=7$ (all eigenvalues are equal to 7)

- $e^{tA} = e^{7t}\left\{I + t(A-7I) + \frac{t^2}{2!}(A-7I)^2 + \frac{t^3}{3!}(A-7I)^3 + \frac{t^4}{4!}(A-7I)^4 + \cdots\right\}$

$= e^{7t}\left\{I + t(A-7I) + \frac{t^2}{2!}(A-7I)^2\right\}$

If A is an $n\times n$ matrix and $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then $(A-\lambda_1 I)^n = 0$ [The Cayley-Hamilton theorem

$= e^{7t}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} & 1 & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix}\right\}$

$= e^{7t}\begin{bmatrix} 1-\frac{3}{2}t+\frac{1}{2}t^2 & t-\frac{1}{2}t^2 & -\frac{1}{2}t+\frac{1}{2}t^2 \\ -t+t^2 & 1-t^2 & t+t^2 \\ \frac{1}{2}t+\frac{1}{2}t^2 & -t-\frac{1}{2}t^2 & 1+\frac{3}{2}t+\frac{1}{2}t^2 \end{bmatrix}$

Remark. In general, if A is an $n\times n$ matrix and $\lambda_1=\lambda_2=\cdots=\lambda_n$, then

$$e^{tA} = e^{\lambda_1 t}\left\{I + t(A-\lambda_1 I) + \frac{t^2}{2!}(A-\lambda_1 I)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A-\lambda_1 I)^{n-1}\right\}.$$