Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients: the method of undetermined coefficients

Xu-Yan Chen

- **Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients:**

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t),$$

where $a_2 \neq 0, a_1, a_0$ are constants, and $f(t)$ is a given function (called the nonhomogeneous term).

- **General solution structure:**

$$y(t) = y_p(t) + y_c(t)$$

where $y_p(t)$ is a particular solution of the nonhomog equation, and $y_c(t)$ are solutions of the homogeneous equation:

$$a_2 y_c''(t) + a_1 y_c'(t) + a_0 y_c(t) = 0.$$

- The characteristic roots: $a_2 \lambda^2 + a_1 \lambda + a_0 = 0$
$\Rightarrow$ The complementary solutions $y_c(t)$.

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- **What is this note about?** **The Method of Undetermined Coefficients:** a method of finding $y_p(t)$, when the nonhomog term $f(t)$ belongs a simple class.

- **Main Idea:** Set up a trial function $y_p(t)$, by copying the function form of $f(t)$.

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- Substitute $y_p(t)$ in the nonhomog eq:

$$\begin{aligned} & 3(ae^{4t})'' + (ae^{4t})' - 2ae^{4t} && = 10e^{4t} \\ = \; & 3(16ae^{4t}) + (4ae^{4t}) - 2ae^{4t} \\ = \; & 50ae^{4t} \end{aligned}$$

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$3y'' + y' - 2y = 10e^{4t}, \quad y(0) = -1, y'(0) = 3.$

- Combine y_c and y_p to get

$$\text{Gen Sols of Nonhomg Eq:} \qquad y(t) = \frac{1}{5}e^{4t} + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}.$$

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- Use initial conditions:

$$y(0) = -1 \quad \Rightarrow \tfrac{1}{5} + C_1 + C_2 = -1$$

$$y'(t) = \tfrac{4}{5}e^{4t} - C_1 e^{-t} + \tfrac{2}{3}C_2 e^{\frac{2}{3}t}, \quad y'(0) = 3 \quad \Rightarrow \tfrac{4}{5} - C_1 + \tfrac{2}{3}C_2 = 3$$

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$$\text{Solve this:} \quad \begin{cases} C_1 = -\frac{9}{5} \\ C_2 = \frac{3}{5} \end{cases}$$

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- The solution of the initial value problem:

$$y(t) = \frac{1}{5}e^{4t} - \frac{9}{5}e^{-t} + \frac{3}{5}e^{\frac{2}{3}t}.$$

Nonhomogeneous Linear Equations:

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t),$$

Towards the Rules of Setting Up the Trial Function:

$f(t)$	$y_p(t)$
ke^{rt}	Ae^{rt}
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- Substitute $y_p(t)$ in the nonhomog eq:

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$$\begin{cases} -11A = 0 \\ 8A = -8 \end{cases} \Rightarrow \text{Impossible!}$$

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The choice of the trial function $y_p(t) = Ate^{-2t}$ was **WRONG!**

Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

- **The correct point of view:**

  $f(t) = -8te^{-2t} =$ (a polynomial of degree one)e^{-2t}.

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- The General Solutions of the Nonhomogeneous Equation:

  $y(t) = y_p(t) + y_c(t) = \left(-\frac{11}{8} - t\right) e^{-2t} + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}$.

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- Substitute $y_p(t)$ in the nonhomog eq:
  $$\begin{aligned} -12t^2 = & \quad 3(2C) + (B + 2Ct) - 2(A + Bt + Ct^2) \\ = & \quad (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2 \end{aligned}$$

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- Compare the coefficients of the two sides:
  $$\begin{cases} -2A + B + 6C = 0 \\ -2B + 2C = 0 \\ -2C = -12 \end{cases} \Rightarrow \begin{cases} A = 21 \\ B = 6 \\ C = 6 \end{cases} \Rightarrow y_p(t) = 21 + 6t + 6t^2$$

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- The General Solutions of the Nonhomogeneous Equation:
  $y(t) = y_p(t) + y_c(t) = 21 + 6t + 6t^2 + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}.$

Nonhomogeneous Linear Equations:

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t),$$

Towards the Rules of Setting Up the Trial Function:

$f(t)$	$y_p(t)$
$p_N(t)$ (a polynomial of deg N)	$A_0 + A_1 t + \cdots + A_N t^N$
ke^{rt}	Ae^{rt}
$p_N(t)e^{rt}$	$(A_0 + A_1 t + \cdots + A_N t^N)e^{rt}$
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$$y_p(t) = A\cos(2t).$$

Example 4: Find a particular solution of $3y'' + y' - 2y = 5\cos(2t)$.

- Complementary solutions $y_c(t)$:
  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1 e^{-t} + C_2 e^{\frac{2}{3}t}$
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- Substitute $y_p(t)$ in the nonhomog eq:

$$\begin{aligned} 5\cos(2t) = &\quad 3[A\cos(2t)]'' + [A\cos(2t)]' - 2A\cos(2t) \\ = &\quad 3(-4A\cos(2t)) - 2A\sin(2t) - 2A\cos(2t) \\ = &\quad -14A\cos(2t) - 2A\sin(2t) \end{aligned}$$

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Wrong!

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The choice of the trial function $y_p(t) = A\cos(2t)$ was **WRONG!**

Example 4 (continued): Find a particular solution of $3y'' + y' - 2y = 5\cos(2t)$.

- The correct trial function:

$$y_p(t) = A\cos(2t) + B\sin(2t).$$

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$$\begin{aligned} 5\cos(2t) = \; & 3[A\cos(2t) + B\sin(2t)]'' + [A\cos(2t) + B\sin(2t)]' \\ & -2[A\cos(2t) + B\sin(2t)] \\ = \; & 3[-4A\cos(2t) - 4B\sin(2t)] + [-2A\sin(2t) + 2B\cos(2t)] \\ & -2[A\cos(2t) + B\sin(2t)] \\ = \; & (-14A + 2B)\cos(2t) + (-2A - 14B)\sin(2t). \end{aligned}$$

Example 4 (continued): Find a particular solution of $3y'' + y' - 2y = 5\cos(2t)$.

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- Compare the coefficients of the two sides:

$$\begin{cases} -14A + 2B = 5 \\ -2A - 14B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{7}{20} \\ B = \frac{1}{20} \end{cases}$$

Example 4 (continued): Find a particular solution of $3y'' + y' - 2y = 5\cos(2t)$.

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$$y_p(t) = A\cos(2t) + B\sin(2t).$$
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- A Particular Solution of the Nonhomogeneous Equation:
$y_p(t) = -\frac{7}{20}\cos(2t) + \frac{1}{20}\sin(2t).$

Nonhomogeneous Linear Equations:

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$$

Towards the Rules of Setting Up the Trial Function:

$f(t)$	$y_p(t)$
$p_N(t)$ (a polynomial of deg N)	$A_0 + A_1 t + \cdots + A_N t^N$
$p_N(t)e^{rt}$	$(A_0 + A_1 t + \cdots + A_N t^N)e^{rt}$
$\left\{ \begin{array}{l} p_N(t)\cos(\omega t) \\ \text{and/or} \quad p_N(t)\sin(\omega t) \end{array} \right.$	$(A_0 + A_1 t + \cdots + A_N t^N)\cos(\omega t)$ $+(B_0 + B_1 t + \cdots + B_N t^N)\sin(\omega t)$
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(to be continued)

Example 5: Find a particular solution of

$$3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t$$

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We give two methods.

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We give two methods.

Method 1:

Solve $3y_1'' + y_1' - 2y_1 = 10e^{4t}$ to get a particular solution $y_1(t)$.

Solve $3y_2'' + y_2' - 2y_2 = -8te^{-2t}$ to get a particular solution $y_2(t)$.

Solve $3y_3'' + y_3' - 2y_3 = -12t^2$ to get a particular solution $y_3(t)$.

Solve $3y_4'' + y_4' - 2y_4 = 5\cos(2t)$ to get a particular solution $y_4(t)$.

Solve $3y_5'' + y_5' - 2y_5 = 17e^{-t}\cos t + 34e^{-t}\sin t$ to get a particular solution $y_5(t)$. (Set $y_5(t) = Ae^{-t}\cos t + Be^{-t}\sin t$)

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A particular solution to the original equation:

$$\begin{aligned} y_p(t) &= y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) \\ &= \tfrac{1}{5}e^{2t} + \left(-\tfrac{11}{8} - t\right)e^{-2t} + 21 + 6t + 6t^2 \\ &\quad -\tfrac{7}{20}\cos(2t) + \tfrac{1}{20}\sin(2t) + \tfrac{7}{2}e^{-t}\cos t - \tfrac{11}{2}e^{-t}\sin t. \end{aligned}$$

Example 5 (continued): Find a particular solution of

$3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t$

Method 2:

- Set a **BIIIIIG** trial function:

$$y_p(t) = A_0 e^{4t} + (A_1 + A_2 t)e^{-2t} + (A_3 + A_4 t + A_5 t^2)$$
$$+[A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t}\cos t + A_9 e^{-t}\sin t),$$

with undetermined coefficients $A_0, A_1, \cdots, A_9$.

Example 5 (continued): Find a particular solution of

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- Substitute this in the original equation.

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- Substitute this in the original equation.
- Compare the coefficients of the two sides
  $\Rightarrow$ Linear equations for $A_0, A_1, \cdots A_9$.

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- Substitute this in the original equation.
- Compare the coefficients of the two sides
  $\Rightarrow$ Linear equations for $A_0, A_1, \cdots A_9$.
- Solve $A_0, A_1, \cdots A_9$.
- Finally obtain the particular solution

$$\begin{aligned} y_p(t) &= \tfrac{1}{5}e^{2t} + \left(-\tfrac{11}{8} - t\right)e^{-2t} + 21 + 6t + 6t^2 \\ &\quad -\tfrac{7}{20}\cos(2t) + \tfrac{1}{20}\sin(2t) + \tfrac{7}{2}e^{-t}\cos t - \tfrac{11}{2}e^{-t}\sin t. \end{aligned}$$

  (Computational details skipped here.)

Example 6: Find general solutions of $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

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- Substitute $y_p(t)$ in the nonhomog eq:
  $$\begin{aligned} 36e^{3t} + 2e^{2t} &= (Ae^{3t} + Be^{2t})'' - (Ae^{3t} + Be^{2t})' - 2(Ae^{3t} + Be^{2t}) \\ &= (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2(Ae^{3t} + Be^{2t}) \\ &= 4Ae^{3t} \end{aligned}$$

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- To find $y_p(t)$, set the trial function
  $$y_p(t) = \cancel{Ae^{3t} + Be^{2t}}. \quad \boxed{\text{Wrong!}}$$
- Substitute $y_p(t)$ in the nonhomog eq:
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- Compare the coefficients of the two sides:
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  The trial function $y_p(t) = Ae^{3t} + Be^{2t}$ was **BAD**!

Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c = C_1 e^{2t} + C_2 e^{-t}$
- The **BAD** trial function: $y_p(t) = Ae^{3t} + Be^{2t}$.

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- **The Reason of the Failure:**

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- **The Reason of the Failure:**
  - When $y_p(t)$ is plugged in the nonhomog eq, we wish the left hand side would match the right hand side $36e^{3t} + 2e^{2t}$.

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  - When $y_p(t)$ is plugged in the nonhomog eq, we wish the left hand side would match the right hand side $36e^{3t} + 2e^{2t}$.
  - The Be^{2t} part of the trial function satisfies the homog eq.

    That is, $(Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0$.

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  - In other words, when plugged in the nonhomog equation, this Be^{2t} produces many terms,
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  - Thus, impossible to balance the two sides of the nonhomog equation.

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  $y_p(t)$ **has overlap(s) with** $y_c(t)$**.**

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- In short, the failure was due to the fact that

  $y_p(t)$ **has overlap(s) with** $y_c(t)$.
- This kind of cases are called *resonance*.

  The term $2e^{2t}$ in $f(t)$ is called a *resonant term*.

Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c(t) = C_1 e^{2t} + C_2 e^{-t}$
- The **NAIVE** trial function: $y_p(t) = Ae^{3t} + Be^{2t}$.
  This failed, since it has a term Be^{2t} overlapping $y_c(t)$.

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- The **CORRECT** trial function: $y_p(t) = Ae^{3t} + Bte^{2t}$.

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- The **CORRECT** trial function: $y_p(t) = Ae^{3t} + Bte^{2t}$.
- **The Method of Correction:** In the naive trial function, multiply the bad term Be^{2t} by t^k, where k is the smallest positive integer to ensure that y_p does not overlap y_c.

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$$\begin{aligned} 36e^{3t} + 2e^{2t} &= (Ae^{3t} + Bte^{2t})'' - (Ae^{3t} + Bte^{2t})' - 2(Ae^{3t} + Bte^{2t}) \\ &= (9Ae^{3t} + 4Be^{2t} + 4Bte^{2t}) - (3Ae^{3t} + Be^{2t} + 2Bte^{2t}) \\ &\quad -2(Ae^{3t} + Bte^{2t}) \\ &= 4Ae^{3t} + 3Be^{2t} \end{aligned}$$

Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c(t) = C_1 e^{2t} + C_2 e^{-t}$
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- Compare the coefficients of the two sides:

$$
\begin{cases} 4A = 36 \\ 3B = 2 \end{cases} \Rightarrow \begin{cases} A = 9 \\ B = 2/3 \end{cases} \Rightarrow y_p(t) = 9e^{3t} + \tfrac{2}{3}te^{2t}
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- The general solutions
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$$\begin{cases} 36a = 9 \\ 2b_0 = 3,\ 6b_1 = -2,\ 12b_2 = -1 \\ 3A + 4B = 1,\ -4A + 3B = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b_0 = \frac{3}{2},\ b_1 = -\frac{1}{3},\ b_2 = -\frac{1}{12} \\ A = \frac{3}{25},\ B = \frac{4}{25} \end{cases}$$

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  $$e^{-t} + \cos(3t) + e^{-t}\sin(3t) \quad = y_p'' + 2y_p' + 10y_p \quad = \cdots\cdots$$
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- The particular solution
  $y_p(t) = \frac{1}{10}e^{-2t} + \frac{1}{37}\cos(3t) + \frac{6}{37}\sin(3t) - \frac{1}{6}te^{-t}\cos(3t).$

Summary of the Method of Undetermined Coefficients

Nonhomog Linear Equations: $a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$

How to set up the trial function?

$f(t)$	$y_p(t)$
$p_N(t)$ (a polynomial of deg N)	$A_0 + A_1 t + \cdots + A_N t^N$
$p_N(t)e^{rt}$	$(A_0 + A_1 t + \cdots + A_N t^N)e^{rt}$
$\begin{cases} \quad p_N(t)\cos(\omega t) \\ \text{and/or} \quad p_N(t)\sin(\omega t) \end{cases}$	$(A_0 + A_1 t + \cdots + A_N t^N)\cos(\omega t)$ $+(B_0 + B_1 t + \cdots + B_N t^N)\sin(\omega t)$
$\begin{cases} \quad p_N(t)e^{rt}\cos(\omega t) \\ \text{and/or} \quad p_N(t)e^{rt}\sin(\omega t) \end{cases}$	$(A_0 + A_1 t + \cdots + A_N t^N)e^{rt}\cos(\omega t)$ $+(B_0 + B_1 t + \cdots + B_N t^N)e^{rt}\sin(\omega t)$

In the case of resonance:

- First pick a naive trial function as in the above table.
- Then multiply the resonant term(s) by t^k, where k is the smallest positive integer to ensure that y_p does not overlap y_c.

Bad News

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- The method of undetermined coefficients does **NOOOOOOT** work, when the equation has variable coefficients :

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Example: $(t-1)y'' - ty' + y = e^{2t}$ cannot be solved by the m.u.c.

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- Even for the equations of constant coefficients:
$$a_2y'' + a_1y' + a_0y = f(t),$$
the m.u.c. does **NOOOOOOT** always work.

  It only works when $f(t)$ is a linear combination of the functions that appear in the table of the last page.

  Examples for which the m.u.c. fails:
$$y'' + y = \tan t, \quad y'' + 2y' + y = e^{t^2}, \quad y'' - y = \frac{1}{1+t}, \quad \cdots$$

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Good News

- There is a more general method, the *variation of parameters*, that can solve any nonhomog linear differential equation, as long as y_c has been provided/prepared.