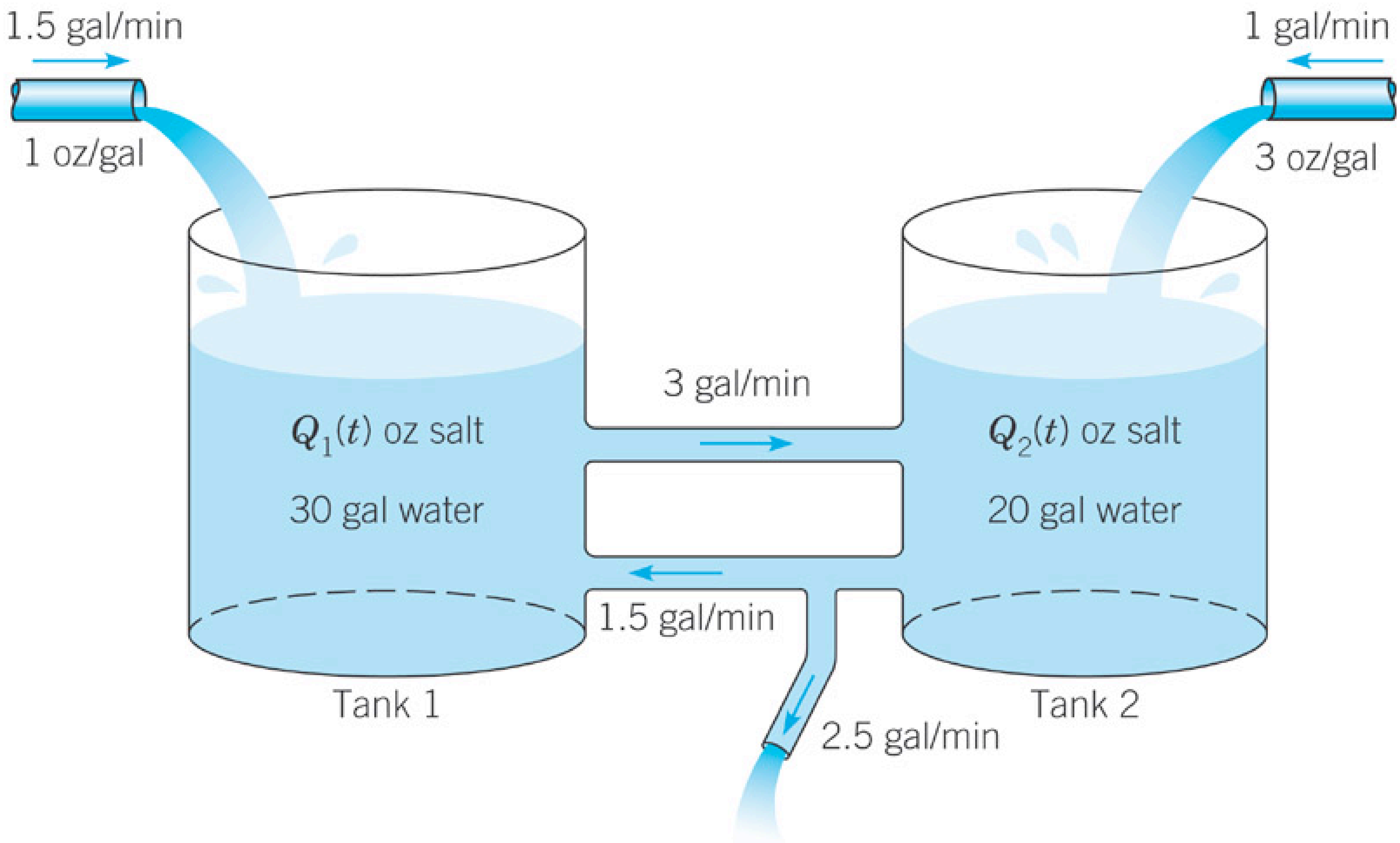
Some Applications of 2-D Systems of Differential Equations

Xu-Yan Chen

Examples:

- Salt in Tanks (a linear system)
- Electric Circuits (a linear system)
- Population Model - Competing Species (a nonlinear system)

Example 1. Salt in Tanks (a linear system)

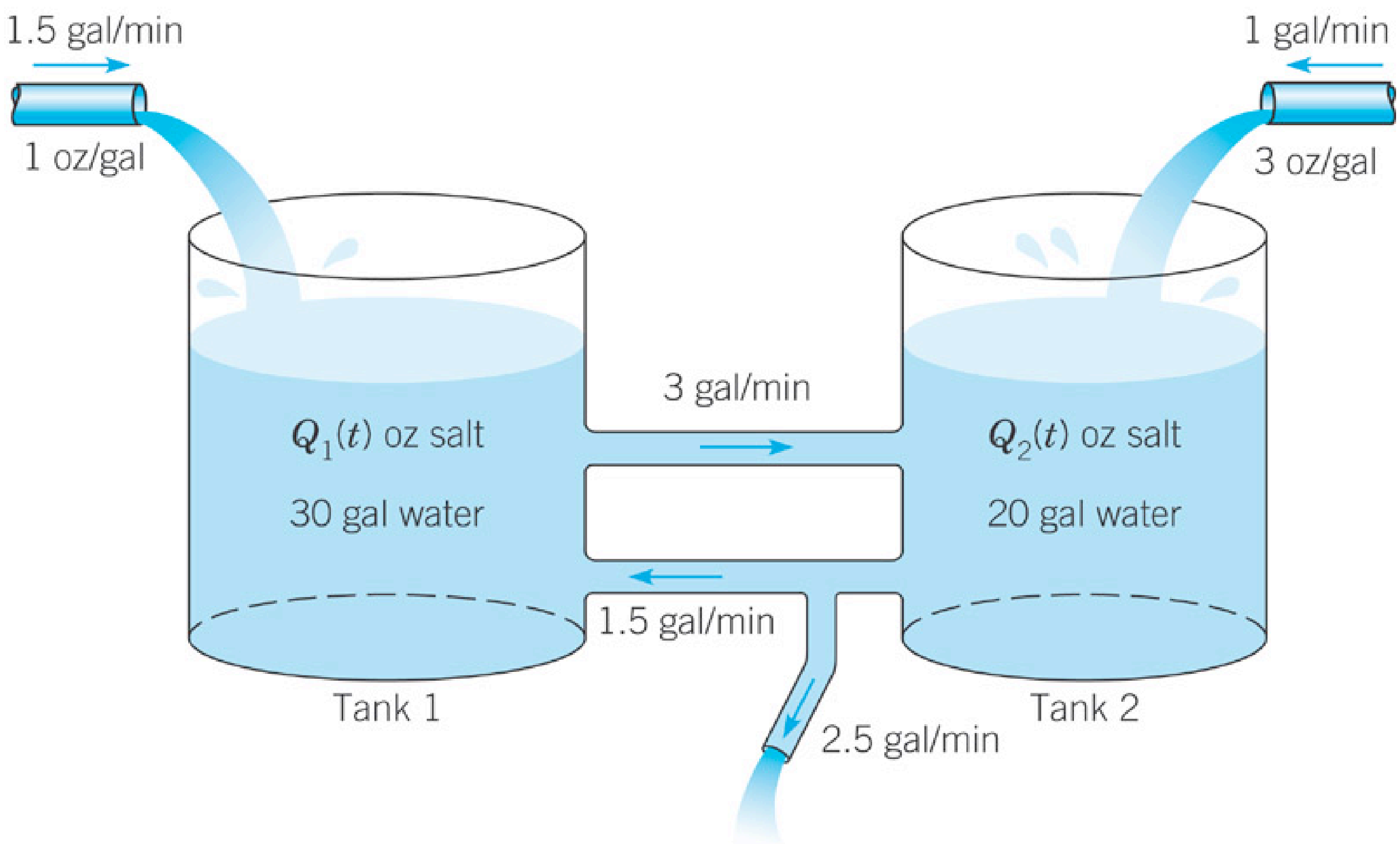


Question: Given initial condition

$Q_1(0) = 55$ oz, $Q_2(0) = 26$ oz,

find $Q_1(t), Q_2(t)$.

Example 1. (continued. Set up equations.)



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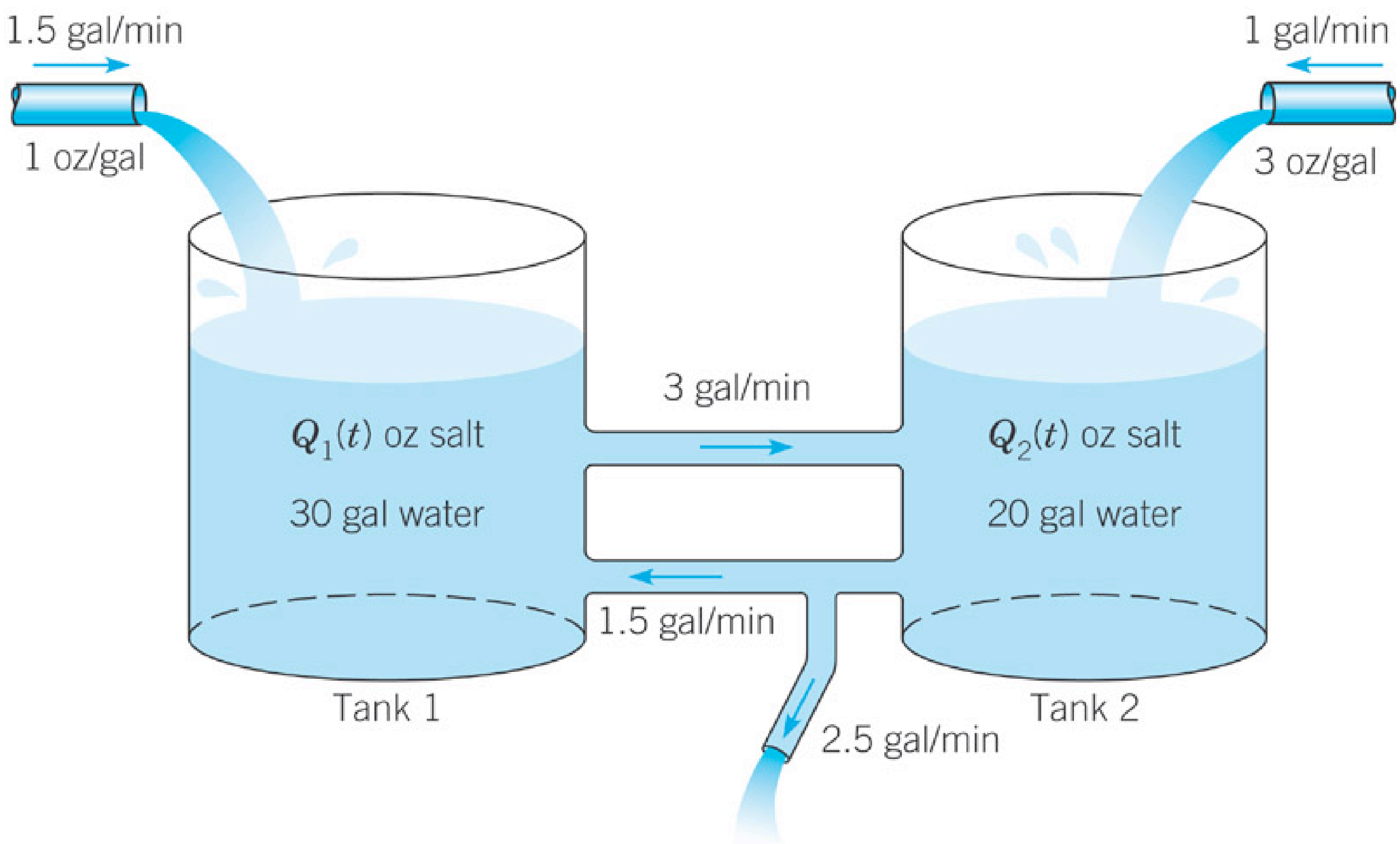
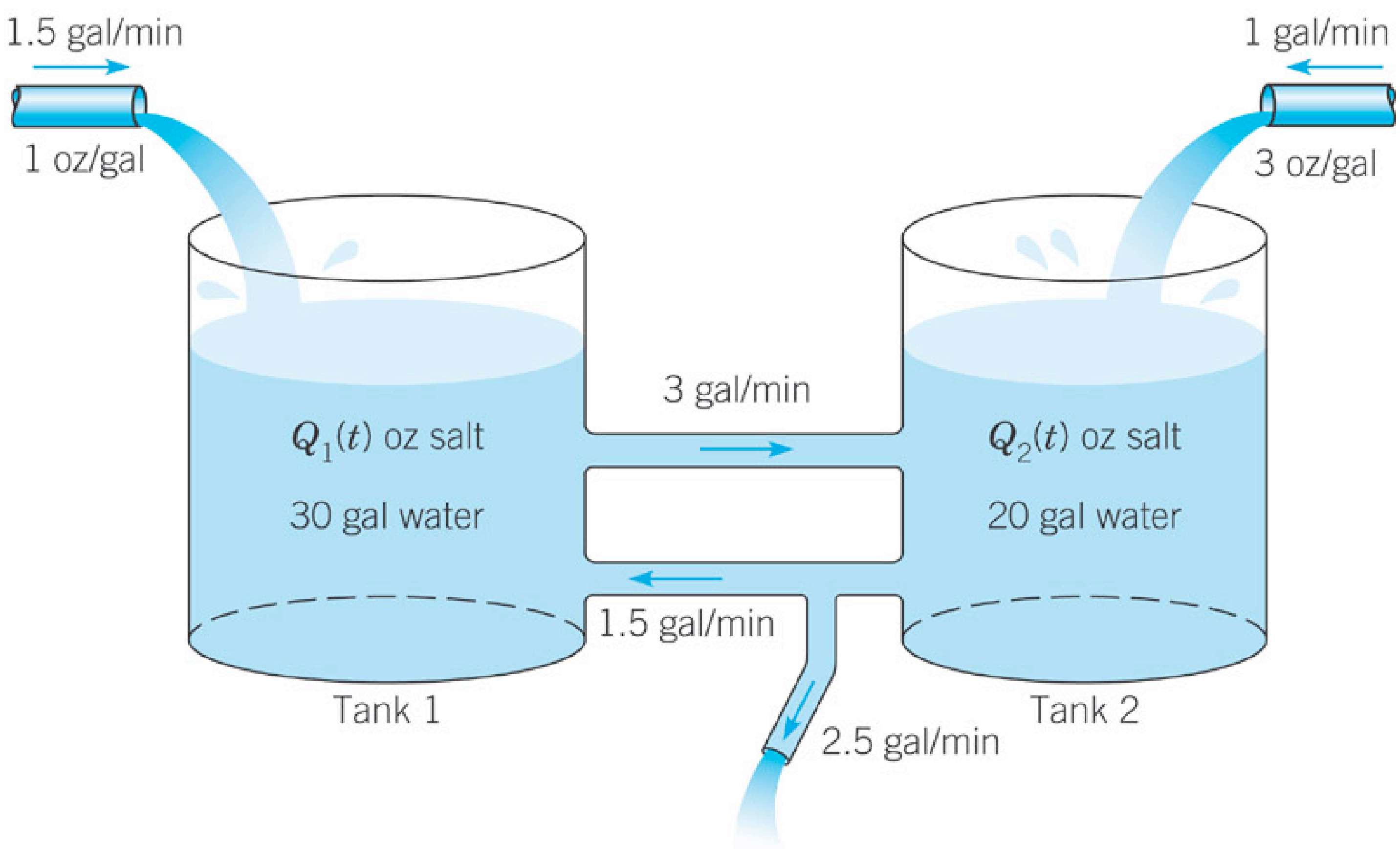


Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) – (rate of salt out)

Example 1. (continued. Set up equations.)



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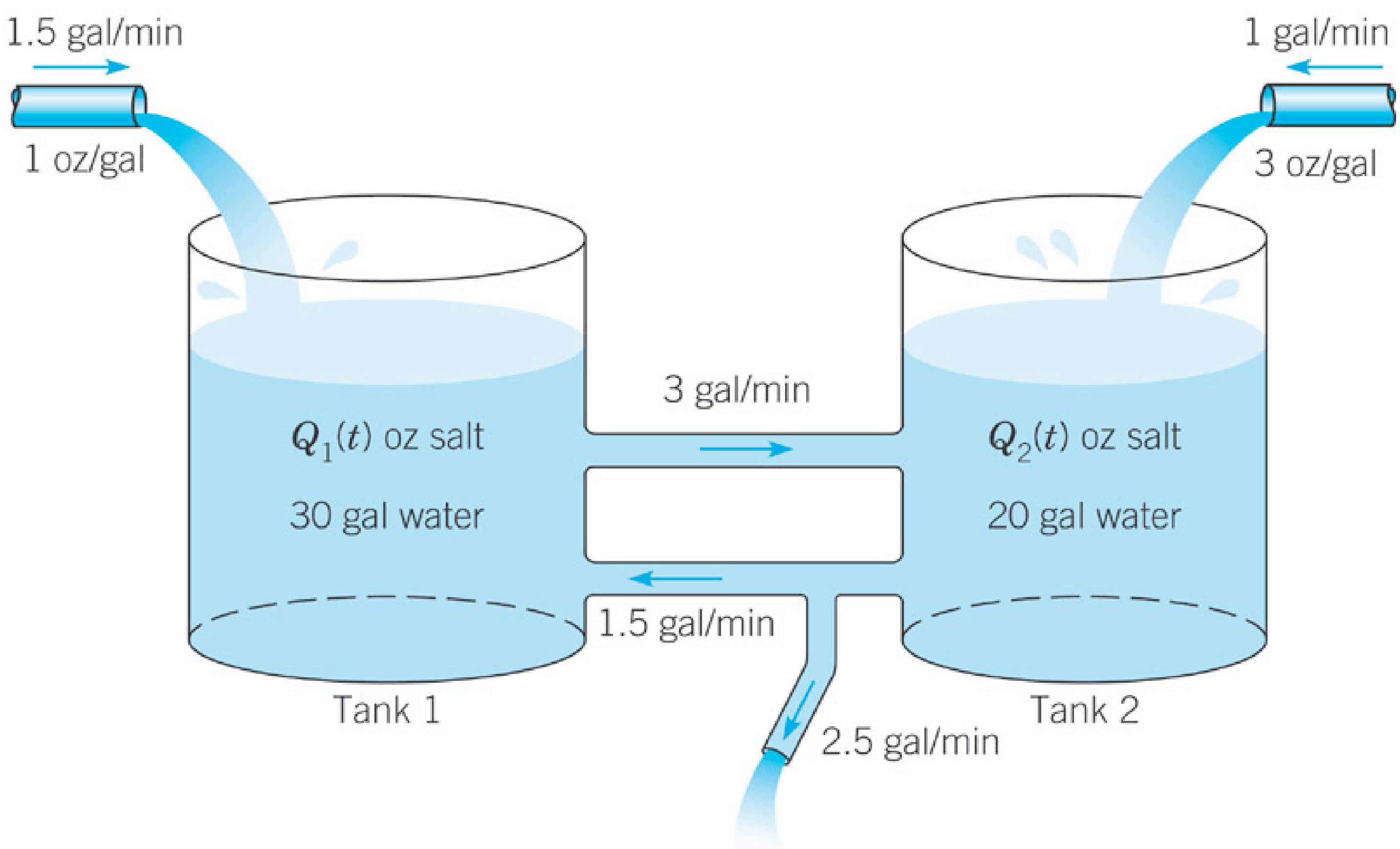
Tank 1: $Q_1'(t) = 1.5 \times 1 \quad + 1.5 \times \dfrac{Q_2(t)}{20} \quad - 3 \times \dfrac{Q_1(t)}{30} \quad (\text{oz}/\text{min})$

Example 1. (continued. Set up equations.)



Differential Equations - based on the conservation law

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Tank 2: $$Q_2'(t) = 1 \times 3 + 3 \times \frac{Q_1(t)}{30} - (1.5 + 2.5) \times \frac{Q_2(t)}{20} \quad (\text{oz/min})$$

Example 1. (continued)

The System of Diff Eqs

$$\begin{cases} Q_1' &= -0.1\ Q_1 + 0.075\ Q_2 + 1.5 \\ Q_2' &= 0.1\ Q_1 - 0.2\ Q_2 + 3 \end{cases}$$

The Matrix Form

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

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Equilibrium: $$\begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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System Recasted

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 - 42 \\ Q_2 - 36 \end{bmatrix}$$

Example 1. (continued)

2-D System

$$\vec{\mathbf{Q}}' = A\left(\vec{\mathbf{Q}} - \vec{\mathbf{a}}\right)$$

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$$A = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix}$$

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Eigenvalues & Eigenvectors of A:

$$\lambda_1 = -0.05, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}; \qquad \lambda_2 = -0.25, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

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General Solutions:

$$\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + C_1 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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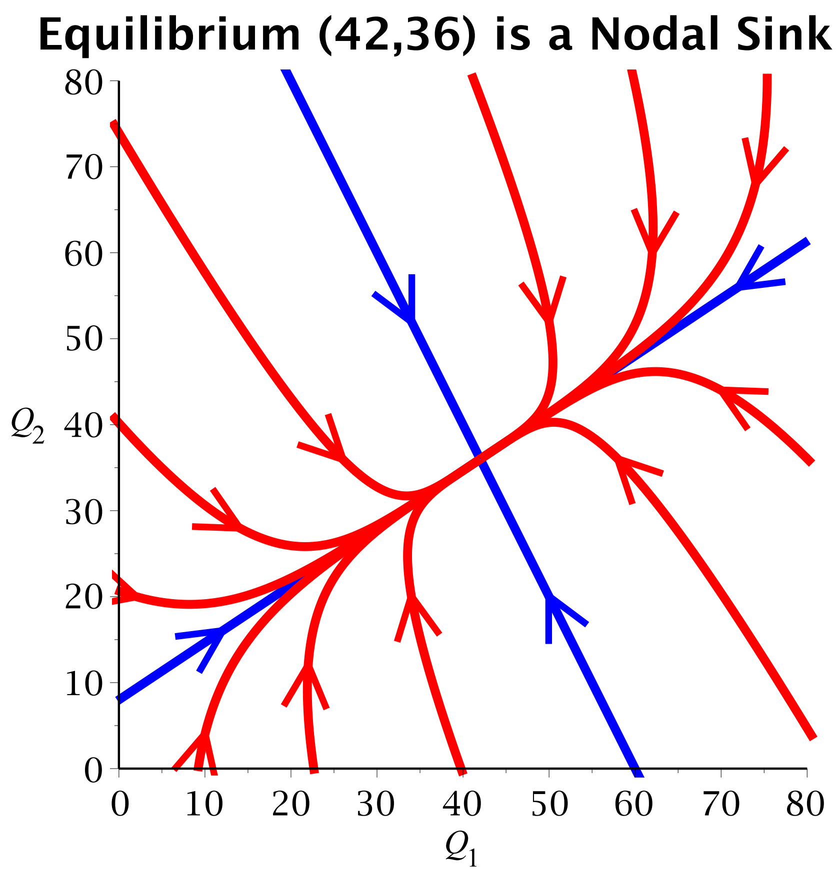

Example 1. (continued. Initial Value Problem)

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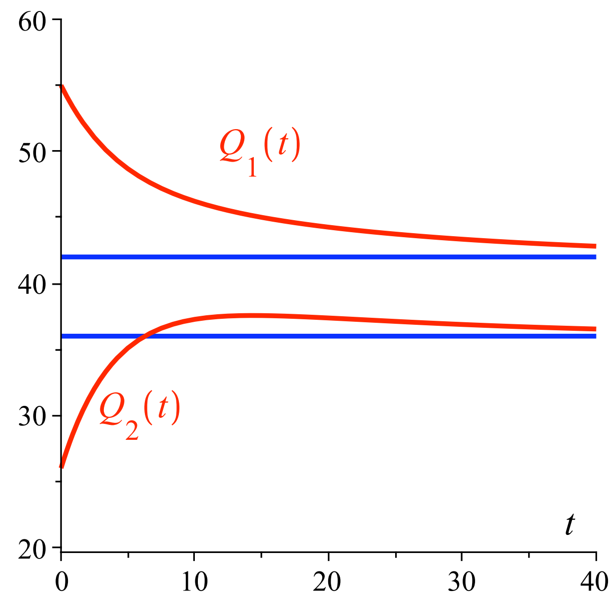
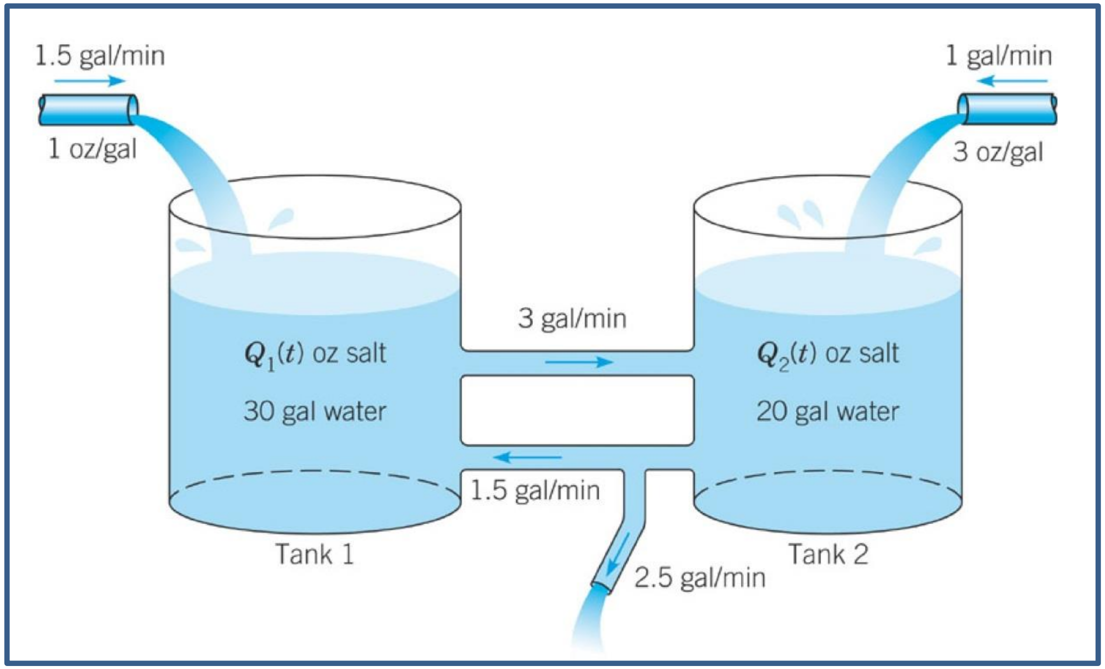


Solution: $\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + 2e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 7e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



Nonhomogeneous Linear Shifted System

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

by a shift:

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Homogeneous Linear System

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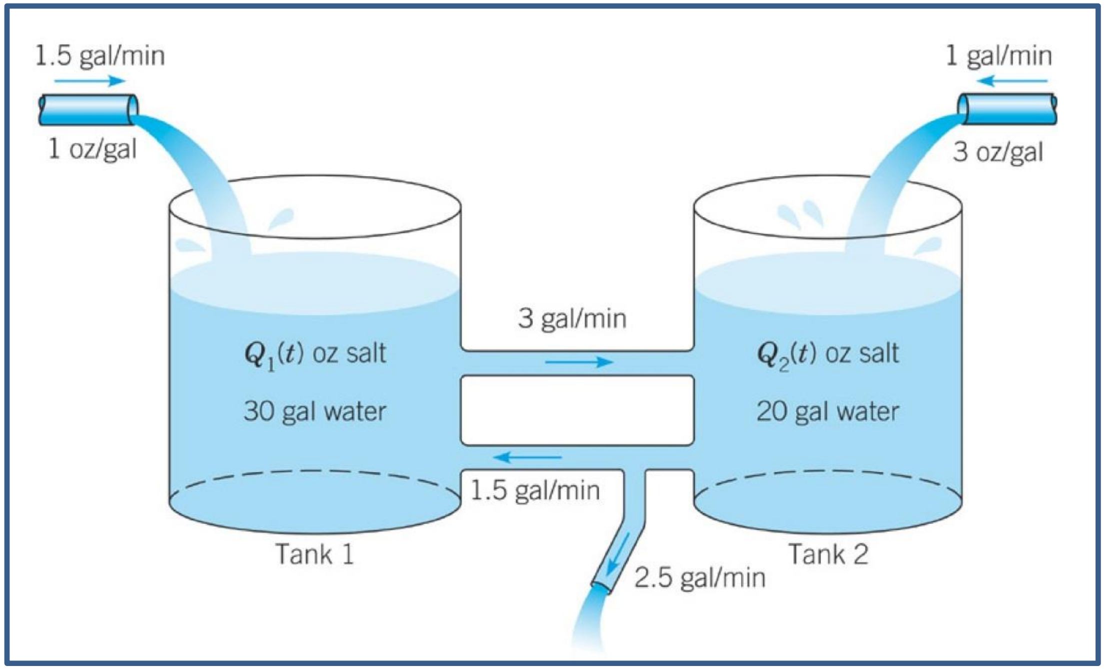

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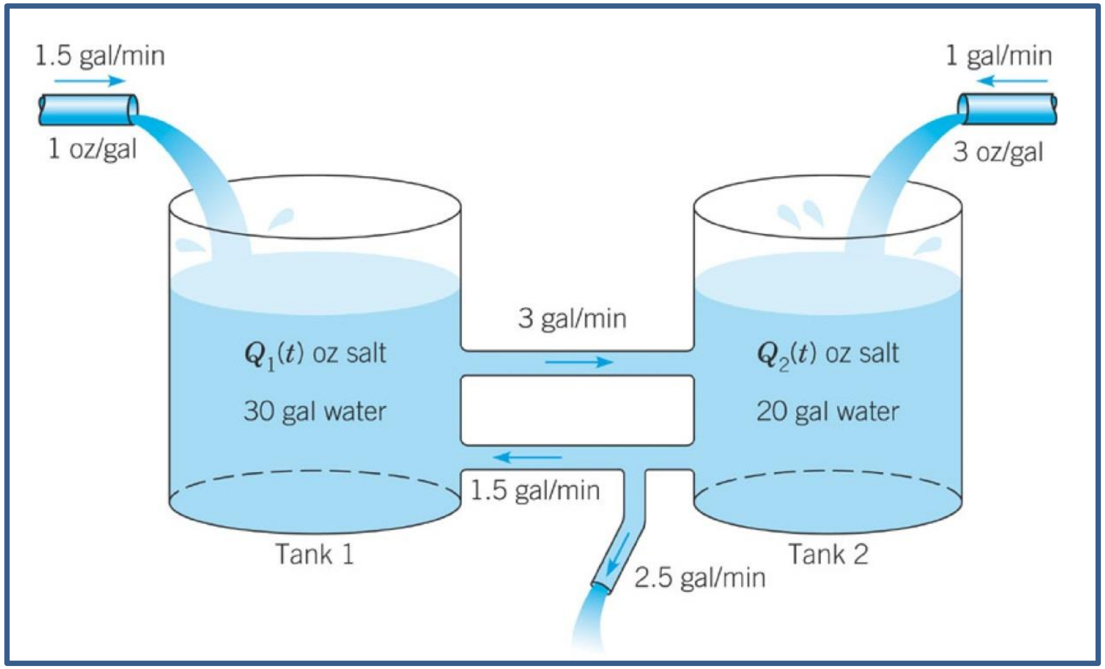
Question: What is the meaning of the homogeneous linear system in the real world?



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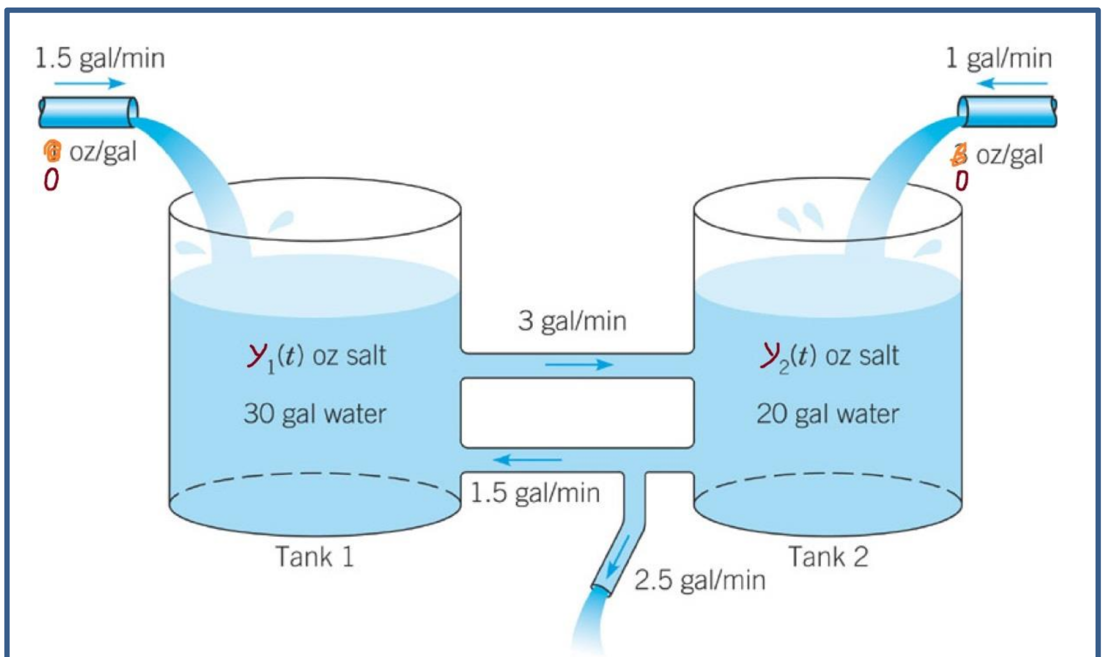

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