

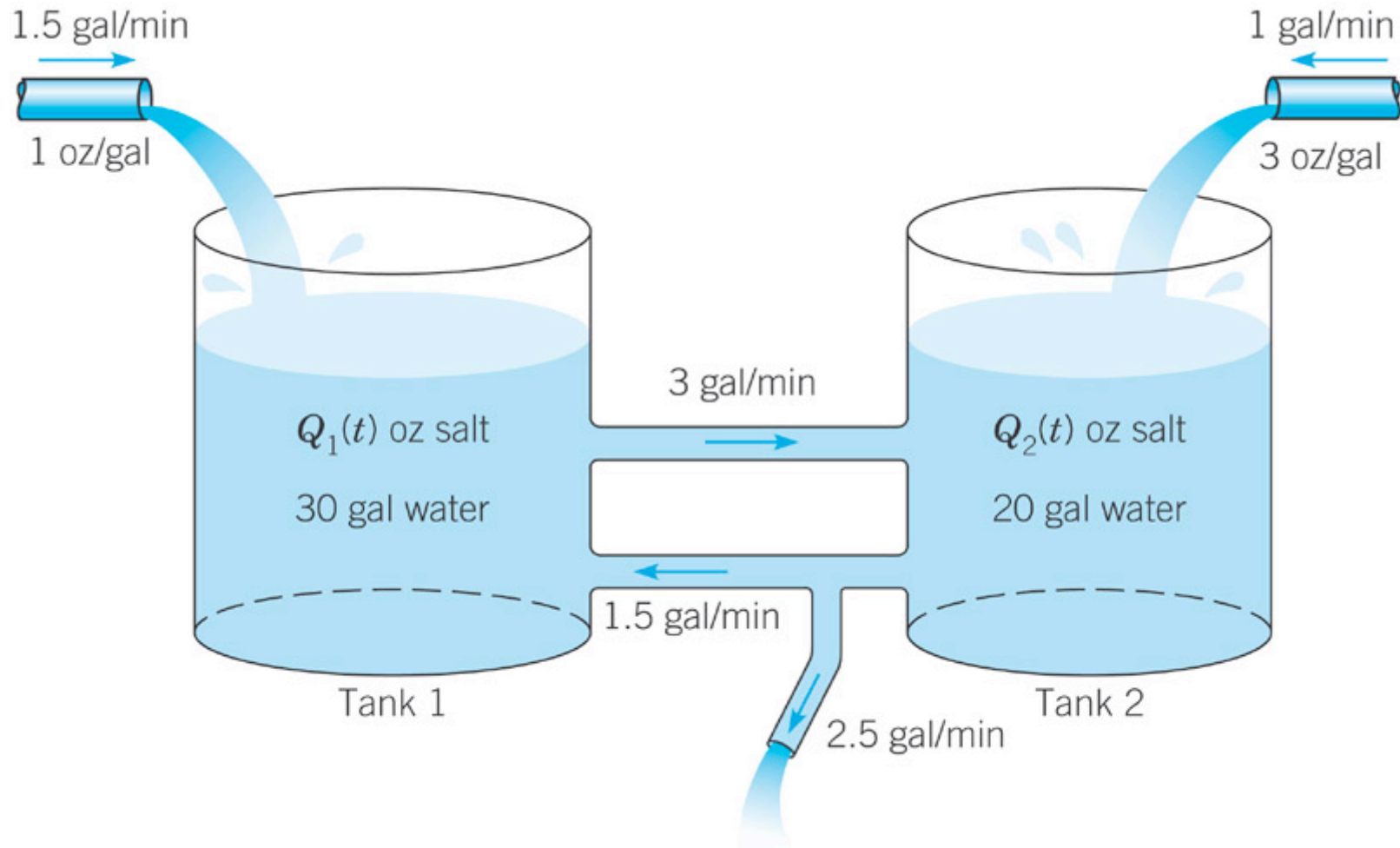
Some Applications of 2-D Systems of Differential Equations

Xu-Yan Chen

Examples:

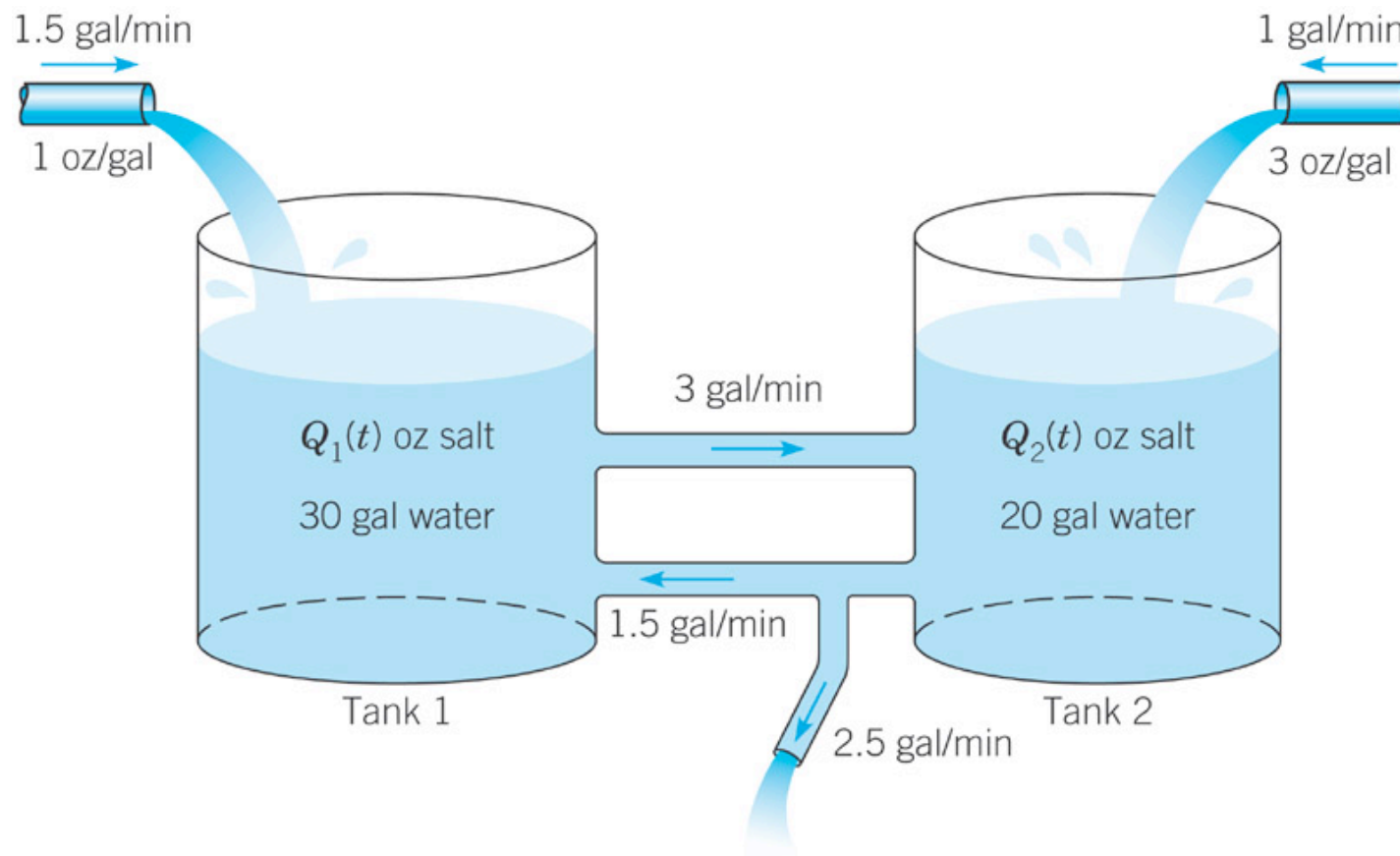
- ▶ Salt in Tanks (a linear system)
- ▶ Electric Circuits (a linear system)
- ▶ Population Model - Competing Species (a nonlinear system)

Example 1. Salt in Tanks (a linear system)

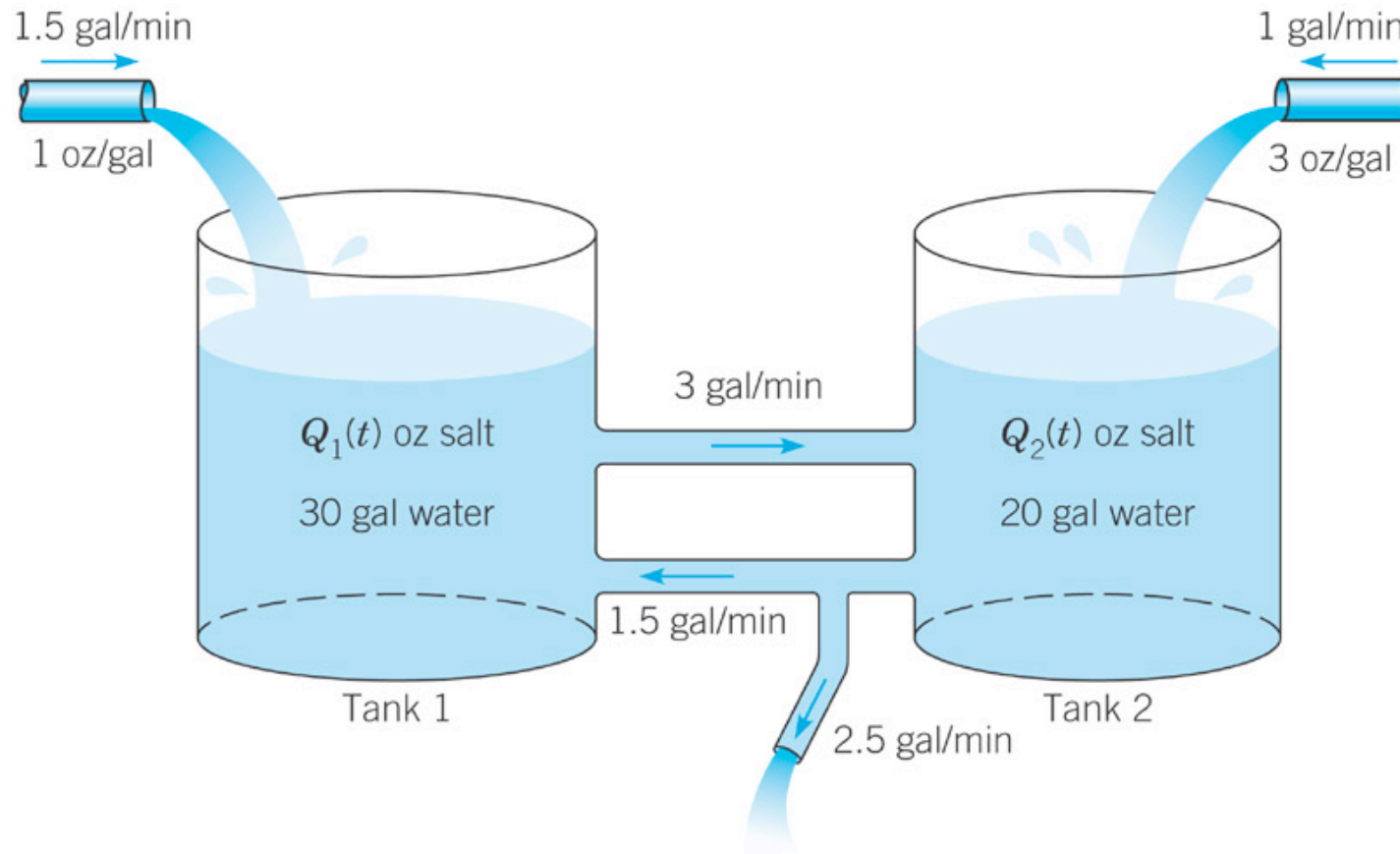


Question: Given initial condition
 $Q_1(0) = 55$ oz, $Q_2(0) = 26$ oz,
find $Q_1(t), Q_2(t)$.

Example 1. (continued. Set up equations.)



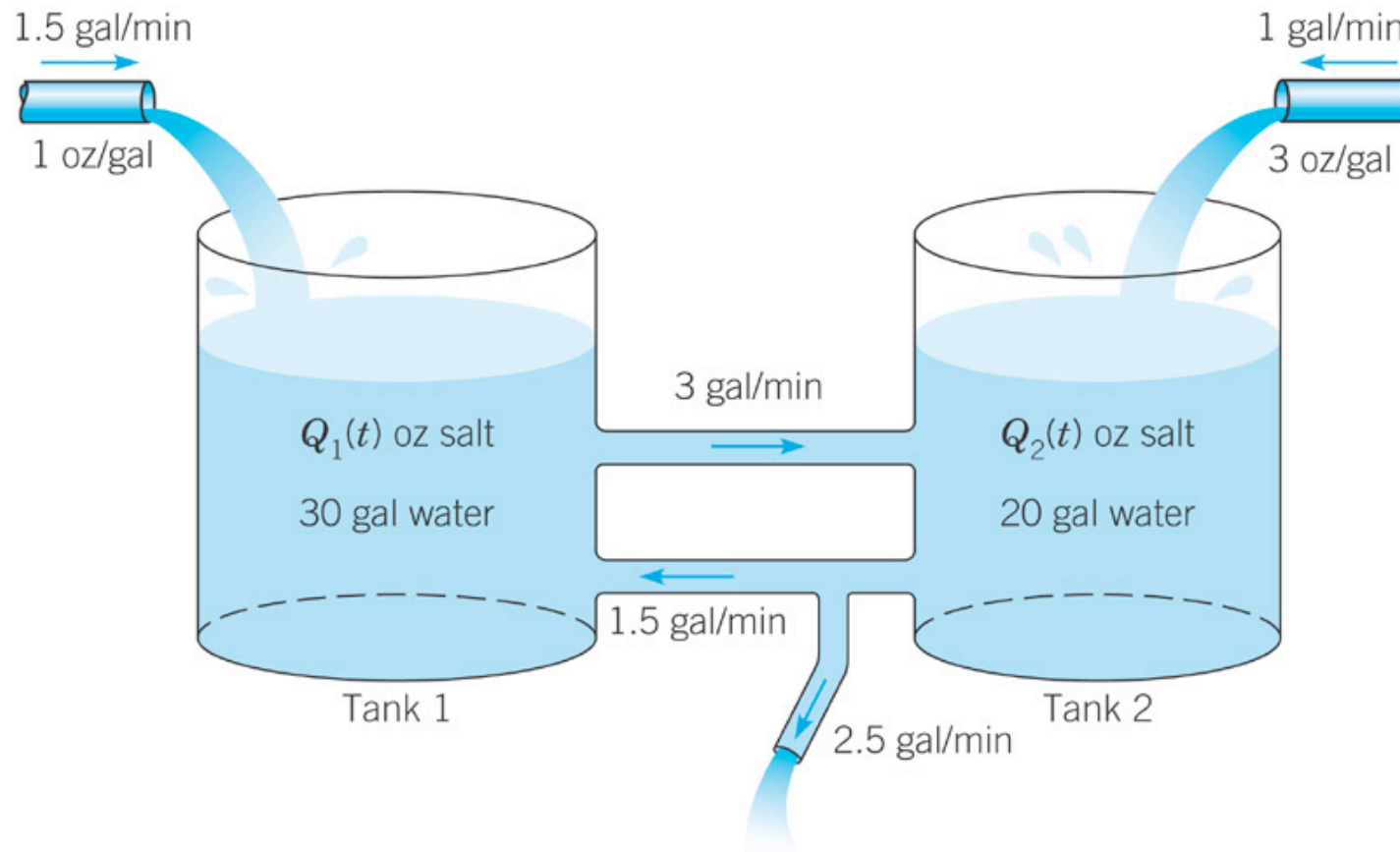
Example 1. (continued. Set up equations.)



Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) - (rate of salt out)

Example 1. (continued. Set up equations.)

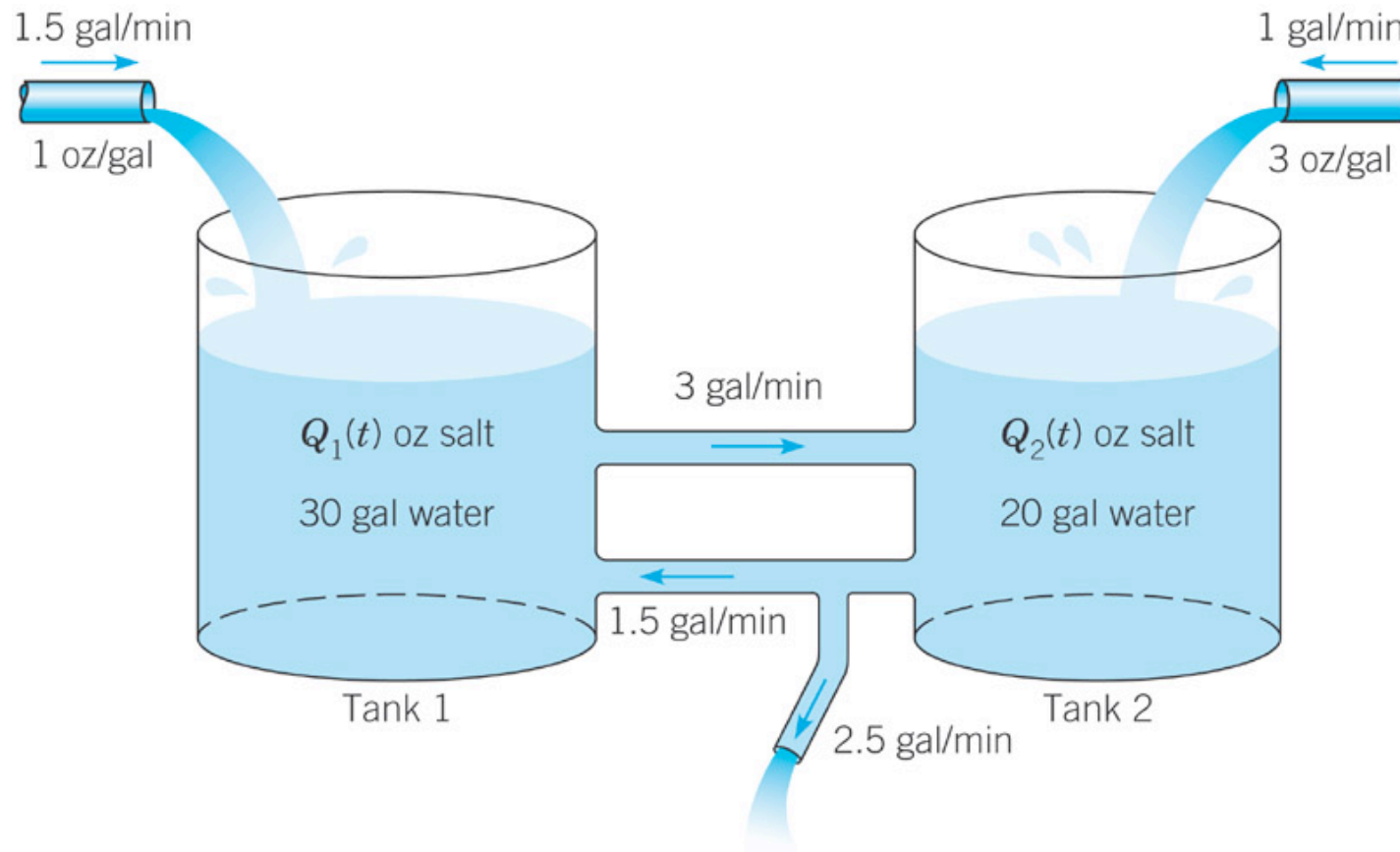


Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) – (rate of salt out)

Tank 1: $Q_1'(t) = 1.5 \times 1 + 1.5 \times \frac{Q_2(t)}{20} - 3 \times \frac{Q_1(t)}{30}$ (oz/min)

Example 1. (continued. Set up equations.)



Differential Equations - based on the conservation law

(rate of change of salt in a tank) = (rate of salt in) – (rate of salt out)

Tank 2:
$$Q_2'(t) = 1 \times 3 + 3 \times \frac{Q_1(t)}{30} - (1.5 + 2.5) \times \frac{Q_2(t)}{20} \quad (\text{oz/min})$$

Example 1. (continued)

$$\begin{array}{l} \text{The System} \\ \text{of Diff Eqs} \end{array} \left\{ \begin{array}{l} Q_1' = -0.1 Q_1 + 0.075 Q_2 + 1.5 \\ Q_2' = 0.1 Q_1 - 0.2 Q_2 + 3 \end{array} \right.$$

$$\text{The Matrix Form} \quad \begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

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$$\text{Equilibrium:} \quad \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Example 1. (continued)

The System of Diff Eqs
$$\begin{cases} Q_1' = -0.1 Q_1 + 0.075 Q_2 + 1.5 \\ Q_2' = 0.1 Q_1 - 0.2 Q_2 + 3 \end{cases}$$

The Matrix Form
$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

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System Recasted
$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 - 42 \\ Q_2 - 36 \end{bmatrix}$$

Example 1. (continued)

2-D System

$$\vec{Q}' = A (\vec{Q} - \vec{a})$$

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The coefficient matrix

$$A = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix}$$

The equilibrium

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$$

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Eigenvalues & Eigenvectors of A :

$$\lambda_1 = -0.05, \quad \vec{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}; \quad \lambda_2 = -0.25, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Example 1. (continued)

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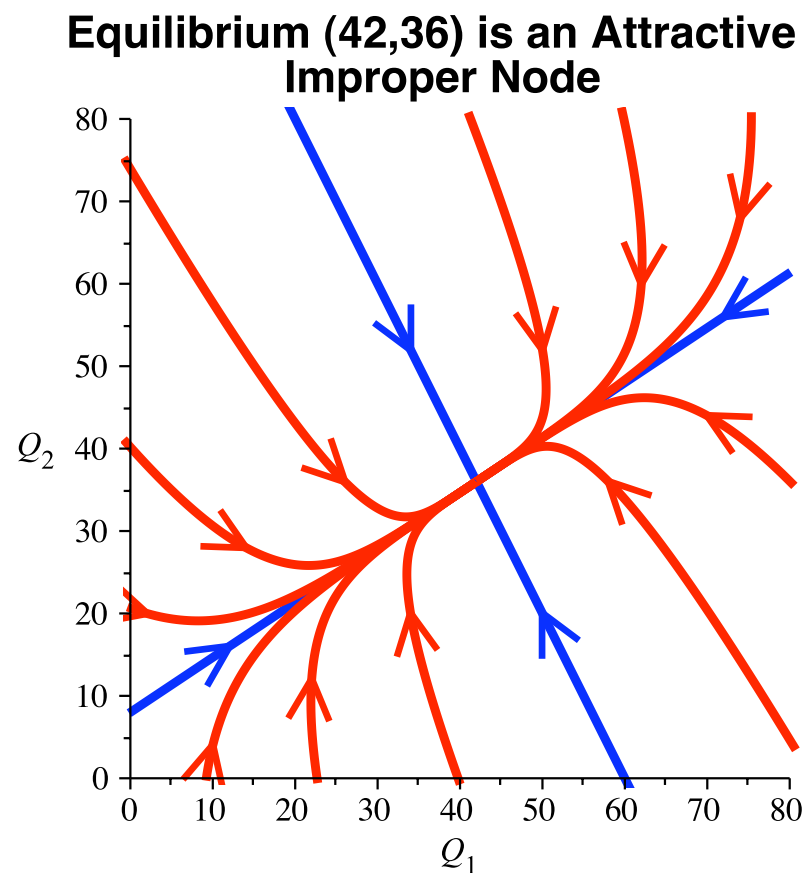
General Solutions:

$$\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + C_1 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 1. (continued)

General Solutions:

$$\vec{Q}(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + C_1 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Example 1. (continued. Initial Value Problem)

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} Q_1 - 42 \\ Q_2 - 36 \end{bmatrix}, \quad \begin{bmatrix} Q_1(0) \\ Q_2(0) \end{bmatrix} = \begin{bmatrix} 55 \\ 26 \end{bmatrix}.$$

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Initial Condition:

$$\begin{bmatrix} 42 \\ 36 \end{bmatrix} + C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 55 \\ 26 \end{bmatrix}$$

$$\implies \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

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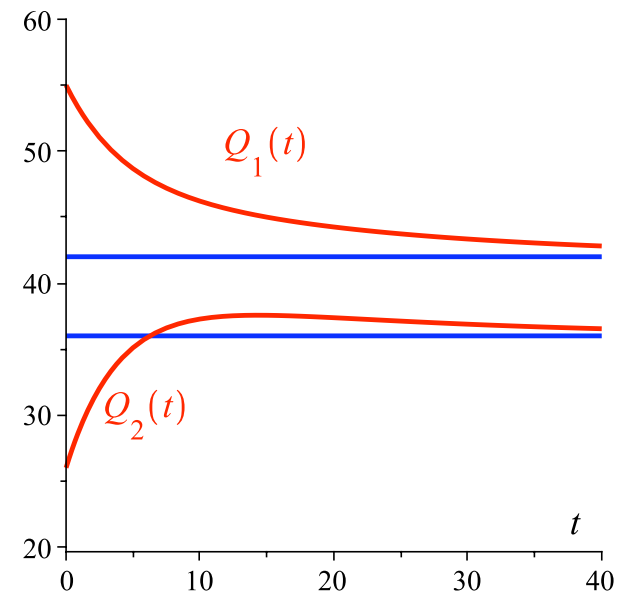
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Solution: $\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix} + 2e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 7e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$