Some Applications of 2-D Systems of Differential Equations

Xu-Yan Chen

Examples:

- Salt in Tanks (a linear system)
- Electric Circuits (a linear system)
- Population Model - Competing Species (a nonlinear system)

Linear Model of Population Dynamics   Malthus (1798)

Diff equation for the population $P(t)$ at time t:

$$P' = rP,$$

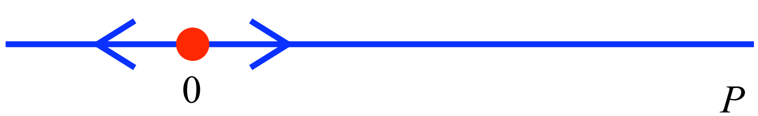
where constant r is the net per capita growth rate:

$$r = b - d = \text{per capita birth rate} - \text{per capita death rate.}$$

Solutions: $P(t) = P(0)e^{rt}$

When $r > 0$,

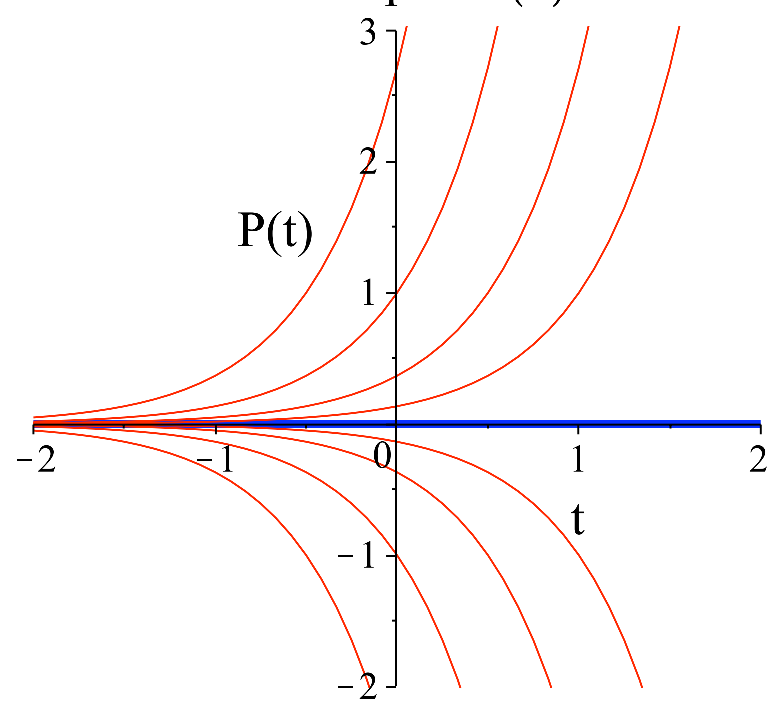
Phase portrait



Equilibrium $P = 0$ is unstable.

Positive solutions $P(t)$ grow exponentially to ∞ as $t \to \infty$.

Solution Graphs $P(t)$ vs t



Logistic Model of Population Dynamics Verhulst (1838)

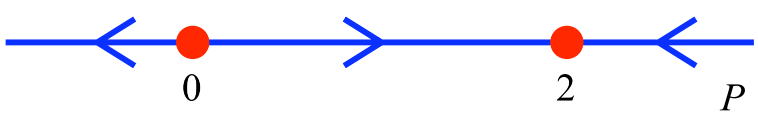
$$P' = rP\left(1 - \frac{P}{K}\right)$$

where r is the net per capita growth rate when $P \approx 0$,
K is the *carrying capacity*.

Solution Formula: $P(t) = \dfrac{KP(0)}{P(0) + [K - P(0)]e^{-rt}}$

Example. $P' = 6P\,(1 - P/2)$ $(r = 6,\ K = 2)$

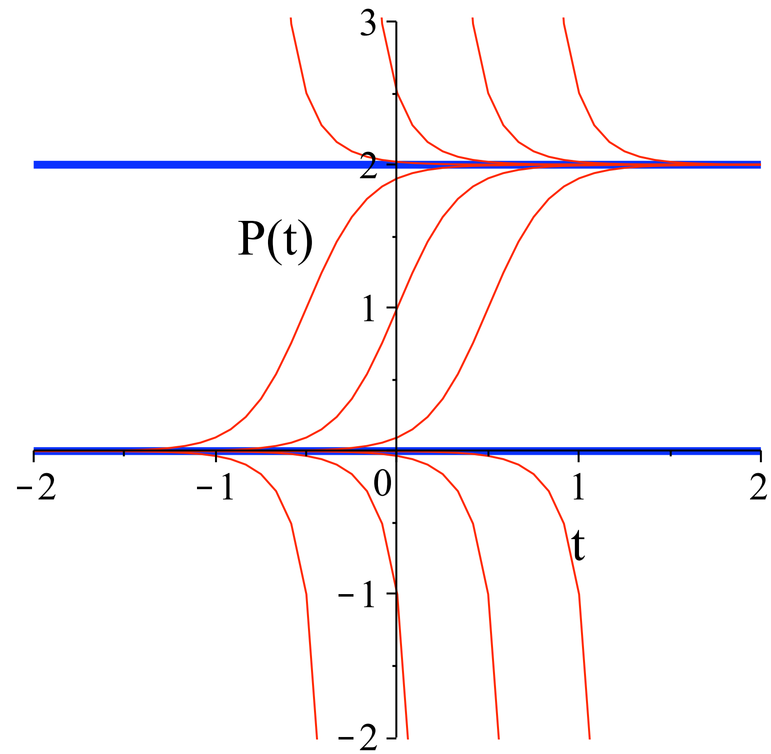
Phase portrait



Equilibrium $P = 0$ is unstable.

Equilibrium $P = K$ is asymp stable.

All positive solutions $P(t)$ converge to K as $t \to \infty$.



Logistic Dynamics of Two Species

If no interactions: $$\begin{cases} x_1' = x_1(6 - 3x_1) \\ x_2' = x_2(3 - 2x_2) \end{cases}$$

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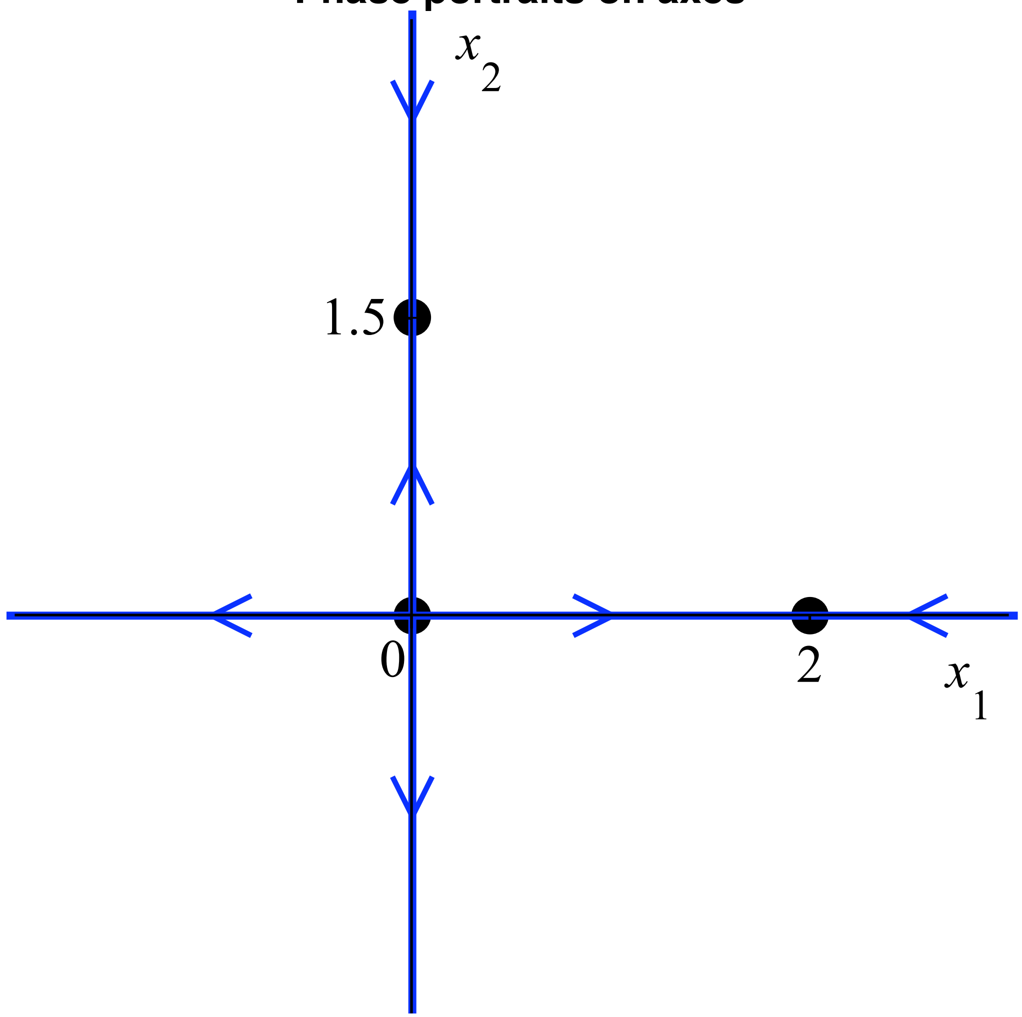
Phase portraits on axes



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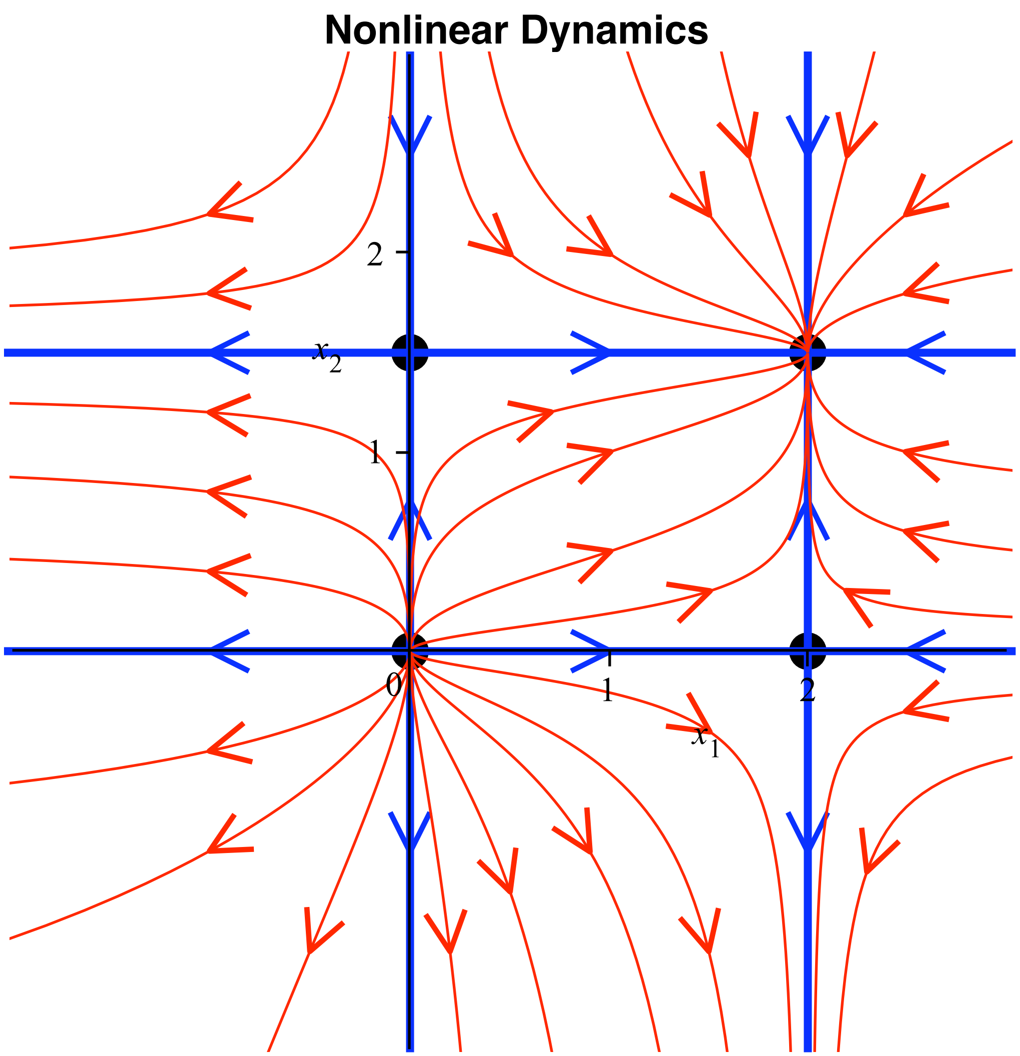

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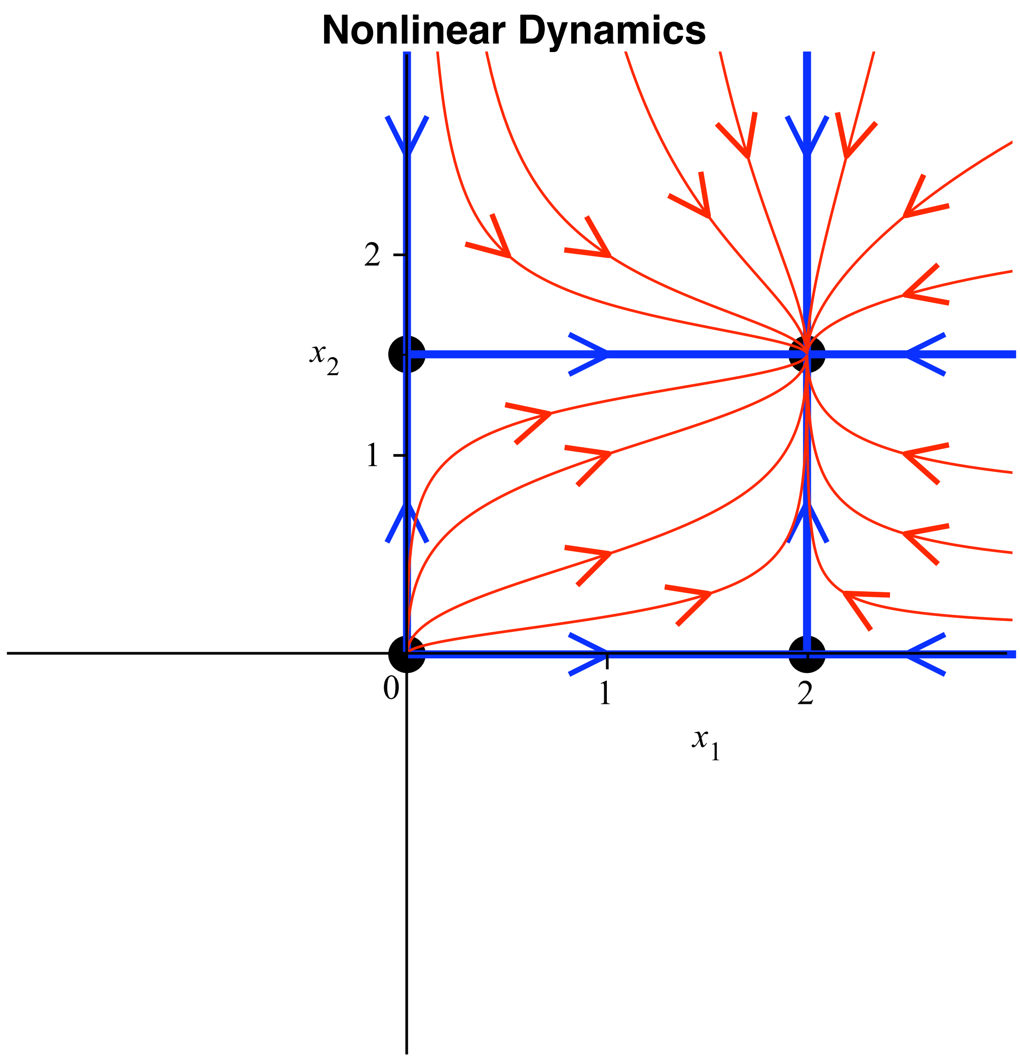

Example 3. Logistic Growth & Competition

Lotka (1925), Volterra (1926), Gause (1934), $\cdots\cdots$

With Competition:

$$\begin{cases} x_1' = x_1(6 - 3x_1 \ \boxed{-2x_2}\ ) \\ x_2' = x_2(3 - 2x_2 \ \boxed{-x_1}\ ) \end{cases}$$

x_2 reduces the growth of x_1

x_1 reduces the growth of x_2

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Basic Questions:

- Find equilibria. (i.e., time independent solutions)
- Construct a linear approximating system near each equilibrium. (use the Jacobian matrix, that is, partial derivatives)
- Study the linear approximating dynamics near the equilibrium. (use eigenvalues & eigenvectors)
- Determine the nonlinear dynamics near the equilibrium. (if eigenvalues are $\neq 0$ & are not purely imaginary, Yes We Can!)

Example 3. (Continued. Find equilibria.)

Competing Species:

$$\begin{cases} x_1' = x_1(6 - 3x_1 - 2x_2) \\ x_2' = x_2(3 - x_1 - 2x_2) \end{cases}$$

$$f_1(x_1, x_2) = x_1(6 - 3x_1 - 2x_2)$$
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Equilibria:

$$\begin{cases} x_1(6 - 3x_1 - 2x_2) = 0 \\ x_2(3 - x_1 - 2x_2) = 0 \end{cases}$$

Example 3. (Continued. Find equilibria.)

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Equilibria:

$$\begin{cases} x_1(6 - 3x_1 - 2x_2) = 0 \\ x_2(3 - x_1 - 2x_2) = 0 \end{cases} \quad \Longrightarrow \text{Separate to four combinations}$$

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Four equilibria:

$$(x_1, x_2) = (0, 0), \quad (2, 0), \quad (0, \tfrac{3}{2}), \quad (\tfrac{3}{2}, \tfrac{3}{4}).$$

Example 3. (continued. Linear Approximation)

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Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$**:**

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Linear Approximating System near equilibrium $(\frac{3}{2}, \frac{3}{4})$:

- Prepare the Jacobian matrix:

$$J = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 6 - 6x_1 - 2x_2 & -2x_1 \\ -x_2 & 3 - x_1 - 4x_2 \end{bmatrix}$$

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- Evaluate J at equilibrium $(x_1, x_2) = (\frac{3}{2}, \frac{3}{4})$:

$$J = \begin{bmatrix} -\frac{9}{2} & -3 \\ -\frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

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Example 3. Linear dynamics near $(0,0)$

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Eigenvalues & Eigenvectors:

$$\lambda_1 = 6, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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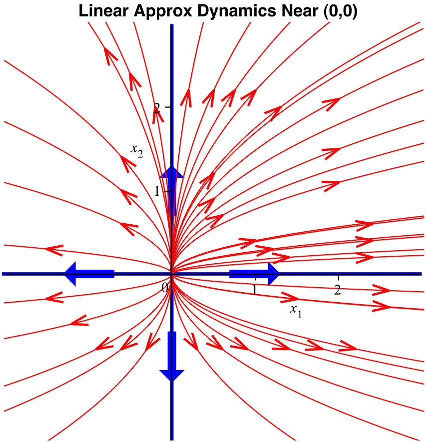

Equilibrium $(0, 0)$ is a nodal source.

Example 3. Linear dynamics near $(2, 0)$

The Linear Approximating System near equilibrium $(2, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 \end{bmatrix}$$

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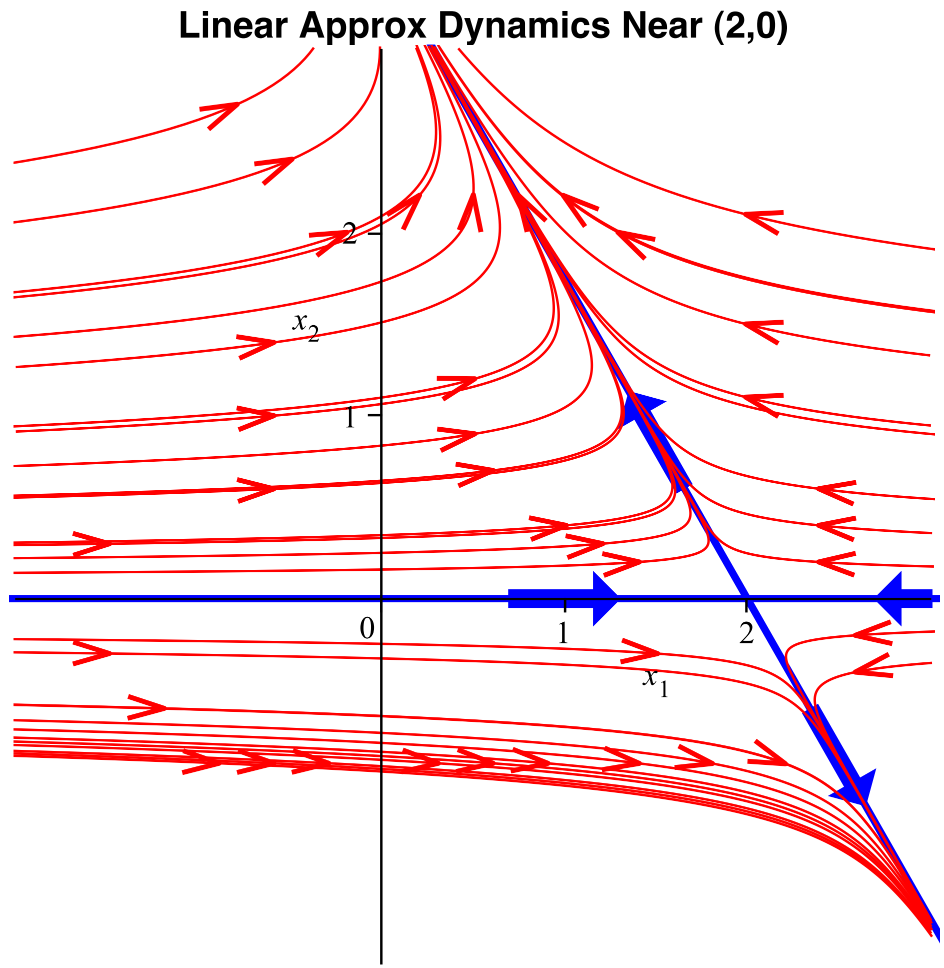

Equilibrium $(2, 0)$ is a saddle

Example 3. Linear dynamics near $(0, \frac{3}{2})$

The Linear Approximating System near equilibrium $(0, \frac{3}{2})$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -\frac{3}{2} & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - \frac{3}{2} \end{bmatrix}$$

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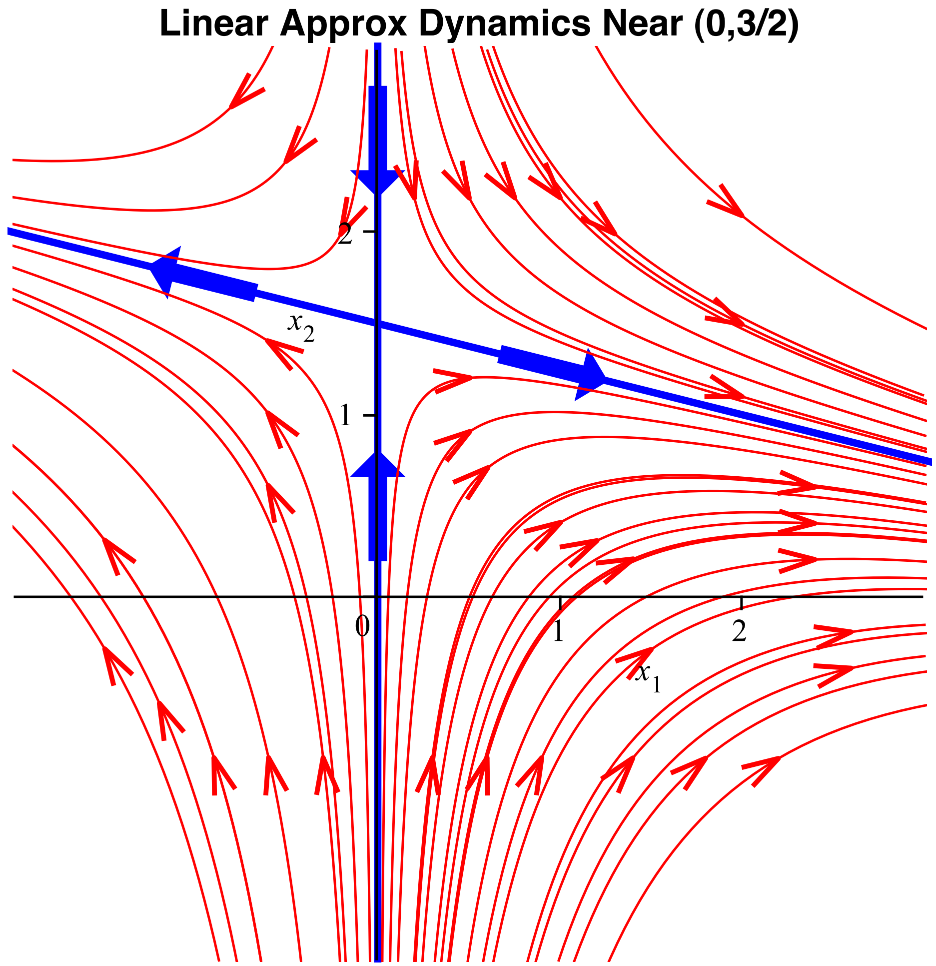

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Eigenvalues & Eigenvectors:

$$\lambda_1 = -3 + \tfrac{3}{2}\sqrt{2} \approx -0.88$$

$$\vec{\mathbf{w}}_1 = \begin{bmatrix} 2 \\ -1 - \sqrt{2} \end{bmatrix}$$

$$\lambda_2 = -3 - \tfrac{3}{2}\sqrt{2} \approx -5.12$$

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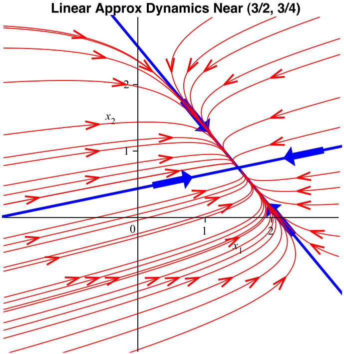
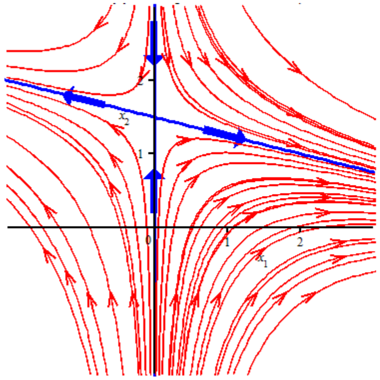
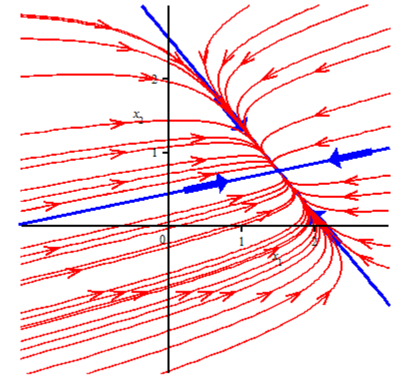


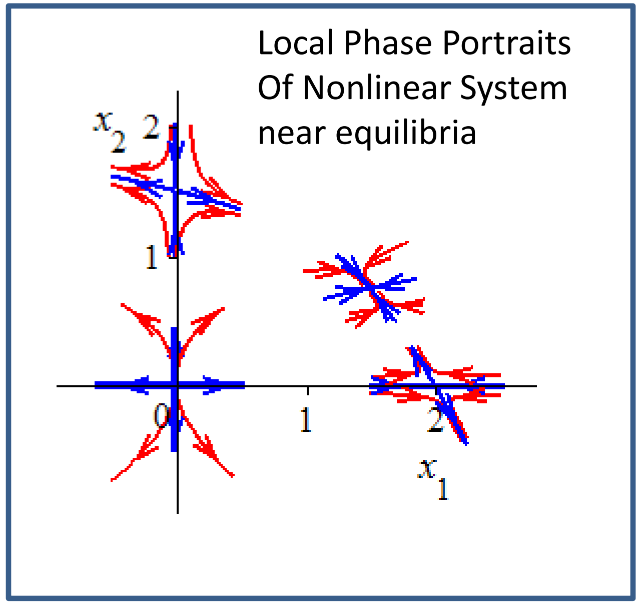
Equilibrium $(\frac{3}{2}, \frac{3}{4})$ is a nodal sink.

Linear Approx Dynamics Near (0,3/2)

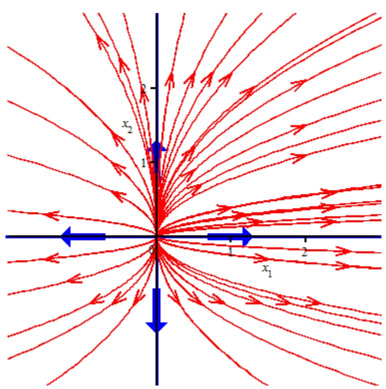


Linear Approx Dynamics Near (3/2,3/4)

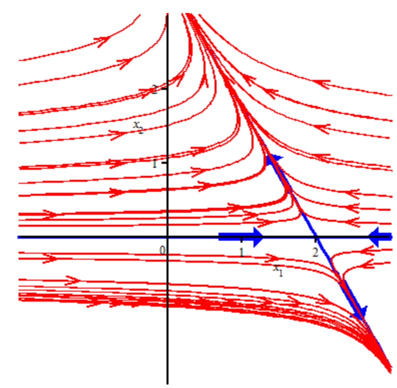




Linear Approx Dynamics Near (0,0)



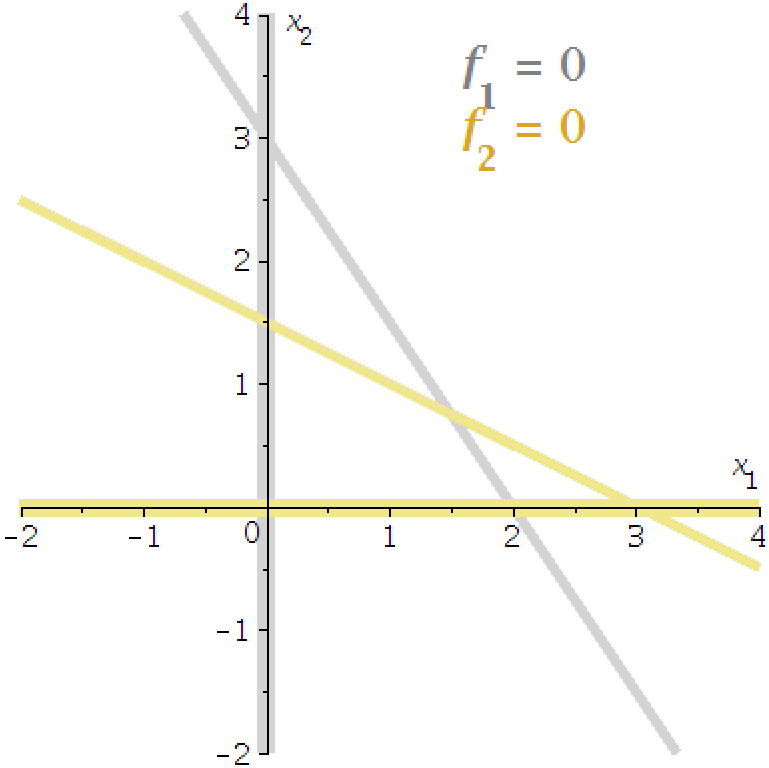
Linear Approx Dynamics Near (2,0)



$$\frac{dx_1}{dt} = x_1(6 - 3x_1 - 2x_2)$$

$$\frac{dx_2}{dt} = x_2(3 - x_1 - 2x_2)$$

Nullclines



$$\frac{dx_1}{dt} = x_1(6 - 3x_1 - 2x_2)$$

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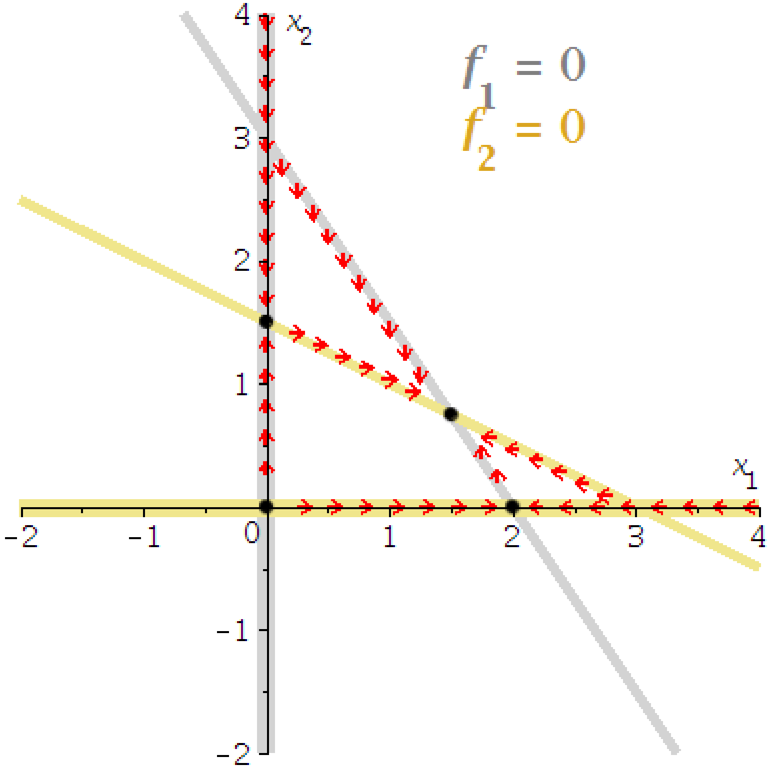
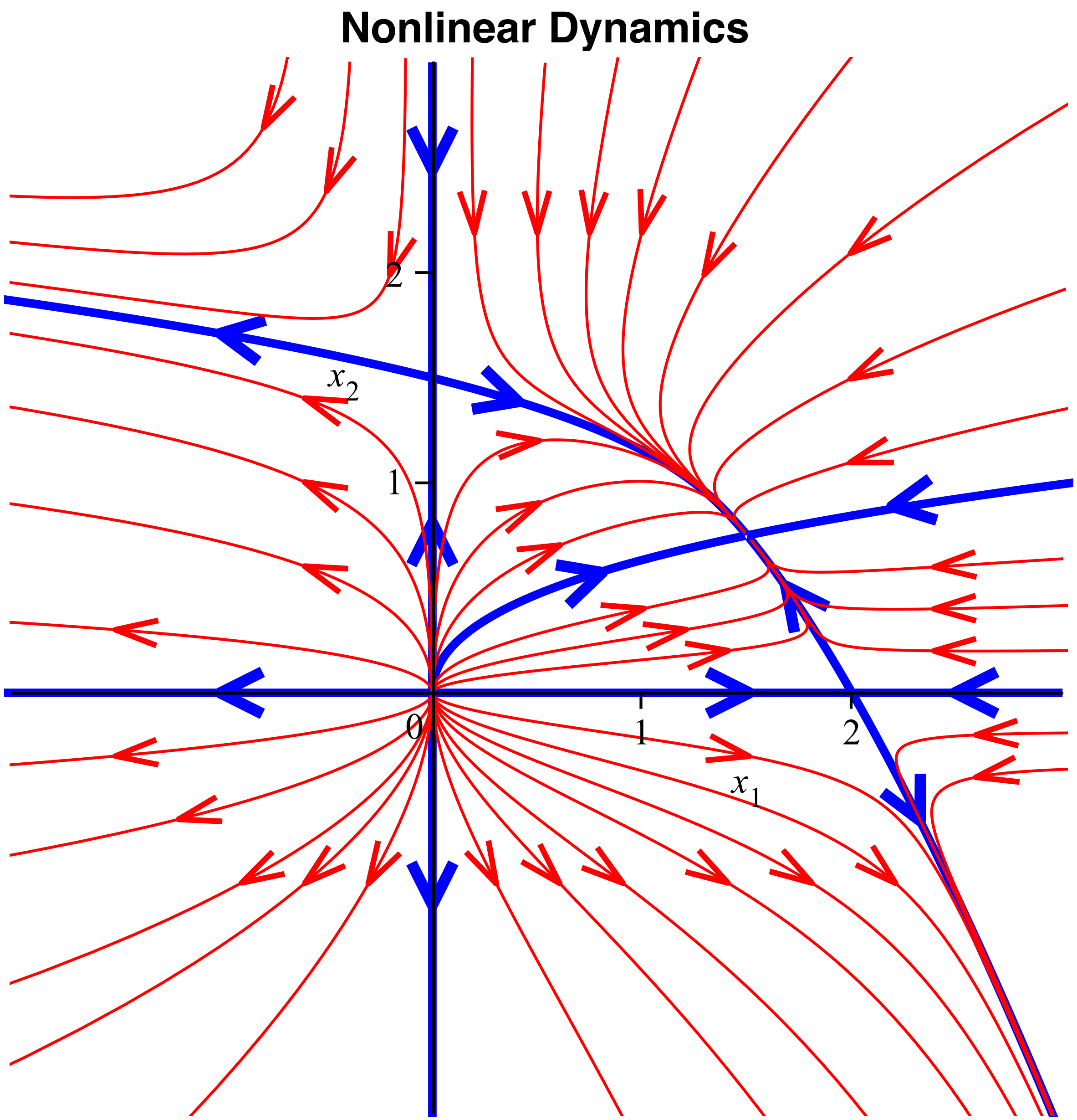
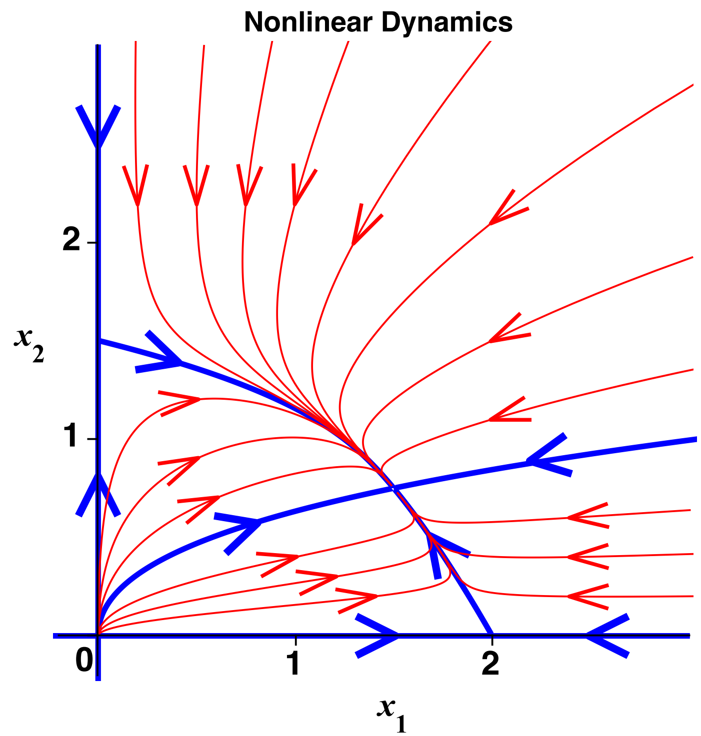
Direction Fields on the Nullclines



Example 3. Global phase portrait

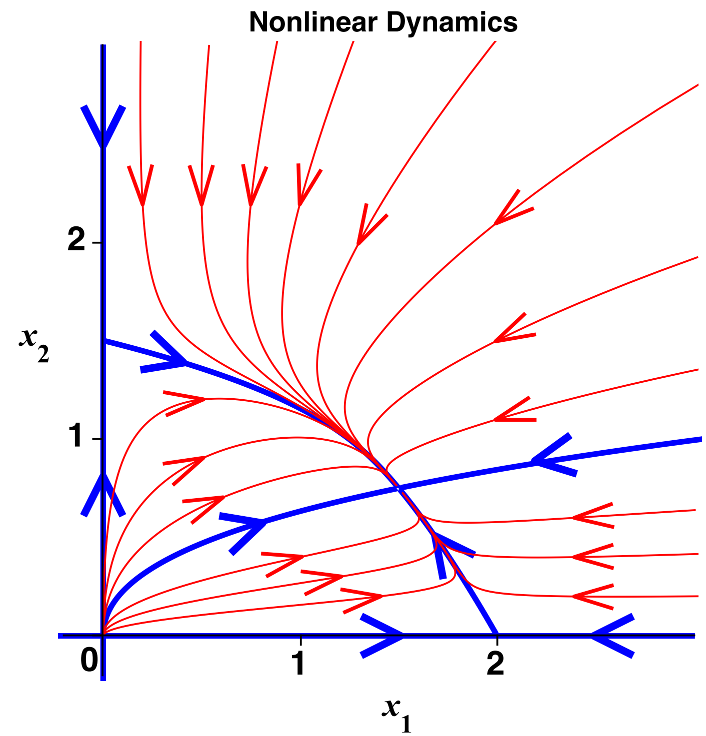


Example 3. Discussion



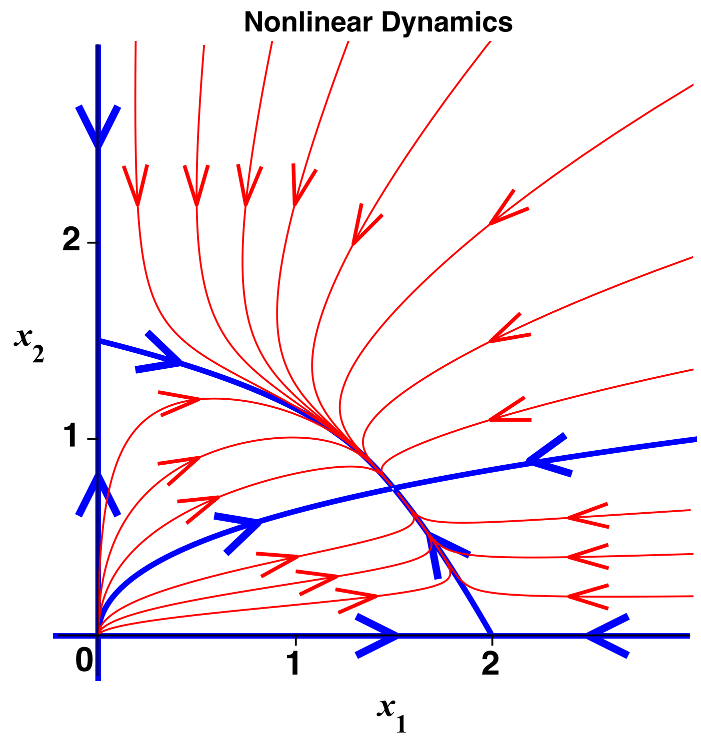
Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.



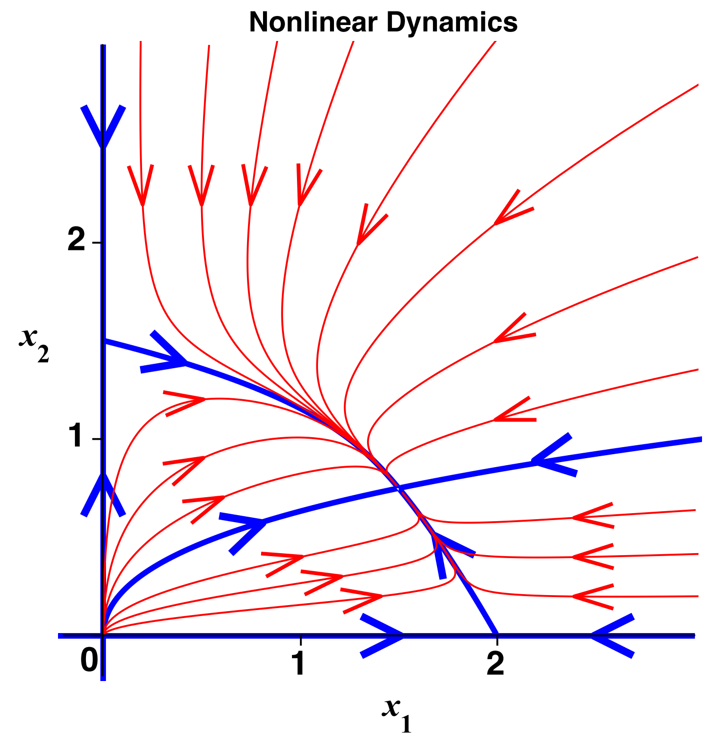
Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.
- The co-existence state $(\frac{3}{2}, \frac{3}{4})$ is asymptotically stable.



Example 3. Discussion

- The survival-extinction states $(2,0)$ and $(0,\frac{3}{2})$ are unstable.
- The co-existence state $(\frac{3}{2},\frac{3}{4})$ is asymptotically stable.
- All positive solutions converge to the co-existence state $(\frac{3}{2},\frac{3}{4})$.



Example 3. Discussion

- The survival-extinction states $(2, 0)$ and $(0, \frac{3}{2})$ are unstable.
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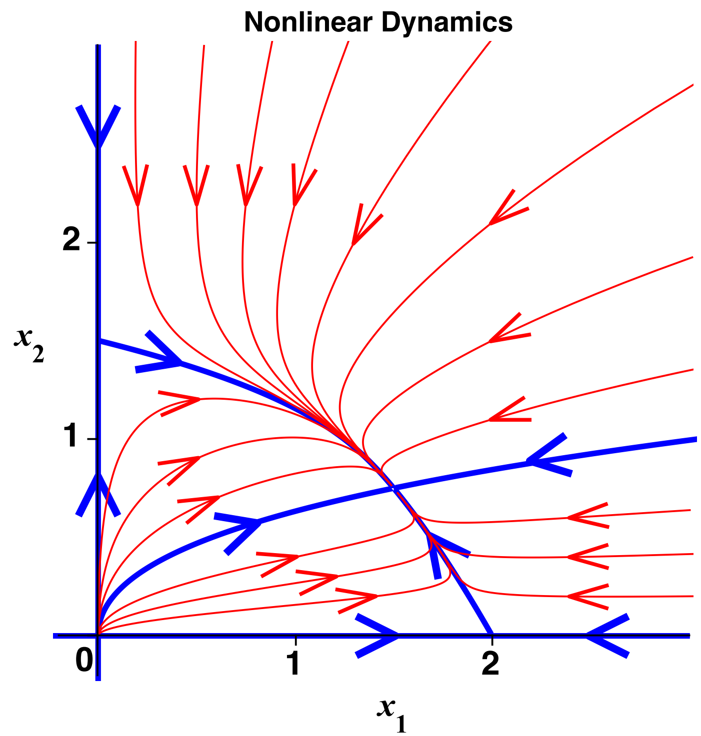
- All positive solutions converge to the co-existence state $(\frac{3}{2}, \frac{3}{4})$.
- A change of the initial populations does not affect the eventual convergence to the co-existence state $(\frac{3}{2}, \frac{3}{4})$.



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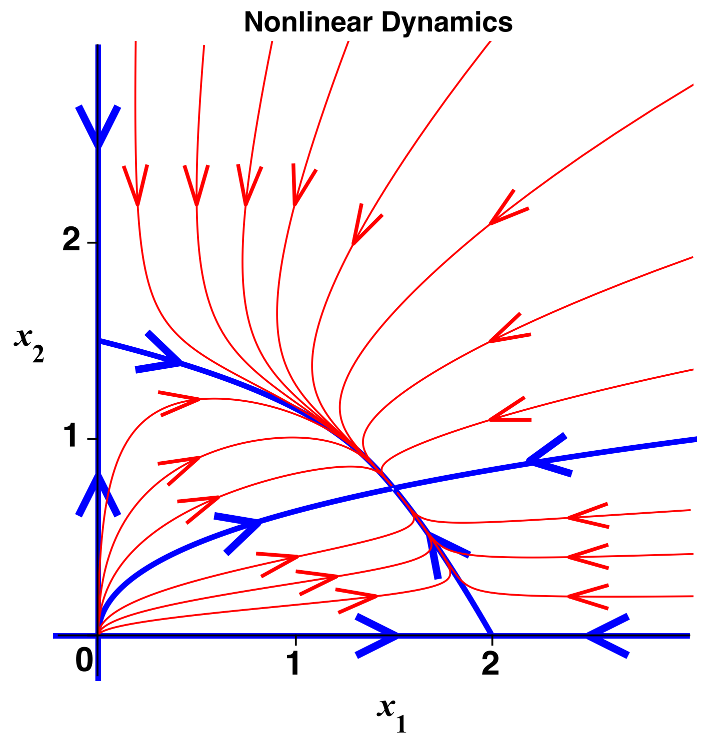

Question: Why is the co-existence stable in this system?

Example 3. Discussion

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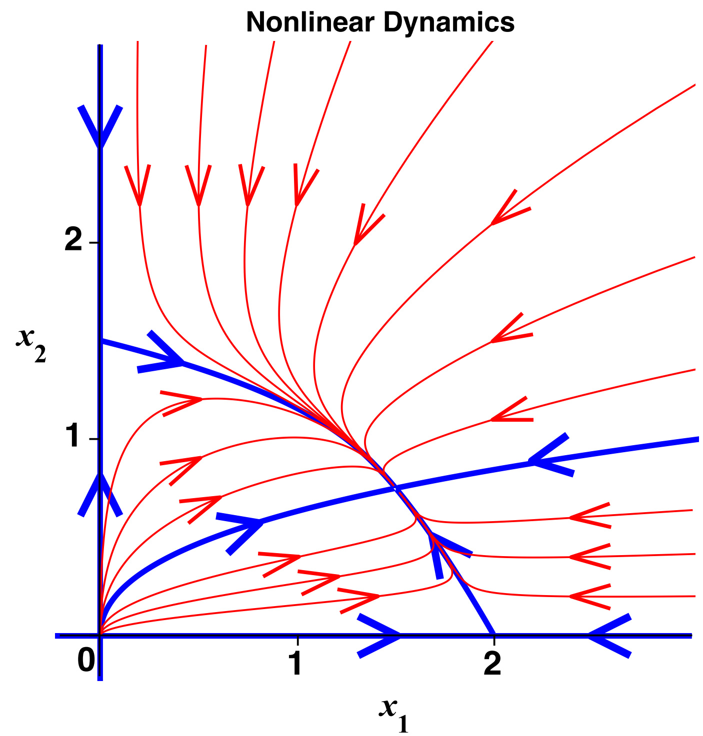
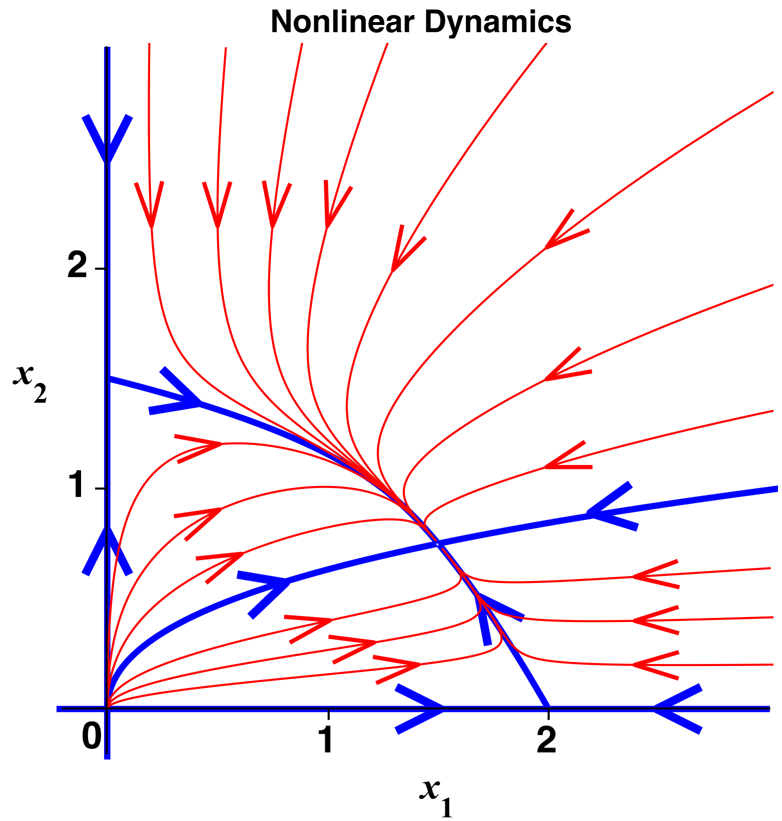


Question: Why is the co-existence stable in this system?

Answer: Weak competition.

Example 3. (continued. Weak competition)

$$\begin{cases} x_1' = x_1(\,6 \;\; -3x_1 \;\; -2x_2\;) \\ x_2' = x_2(\,3 \;\; -x_1 \;\; -2x_2\;) \end{cases}$$



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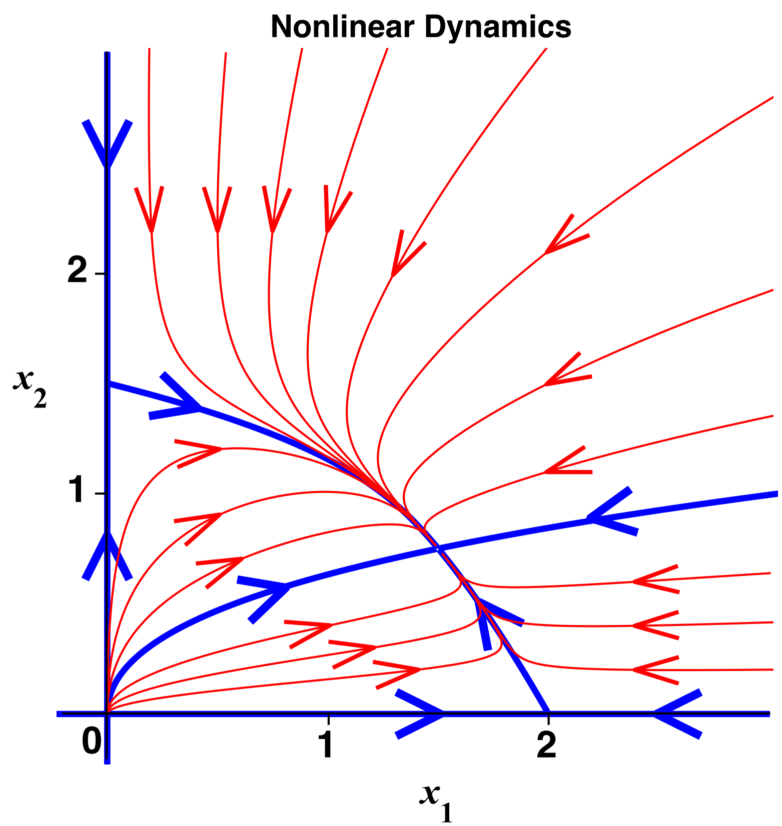
$$\begin{cases} x_1' = x_1(\ 6 \ \ -3x_1 \ \ -2x_2 \ ) \\ x_2' = x_2(\ 3 \ \ -x_1 \ \ -2x_2 \ ) \end{cases}$$

The competition terms

$-2x_2$ and $-x_1$



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The competition terms

$-2x_2$ and $-x_1$

the resource inhibition terms

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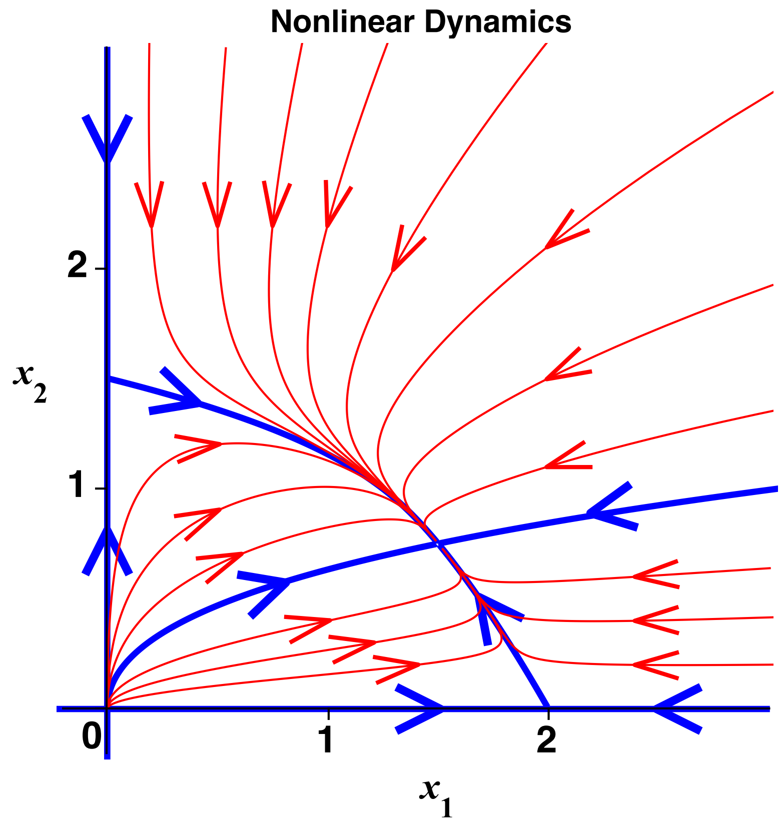

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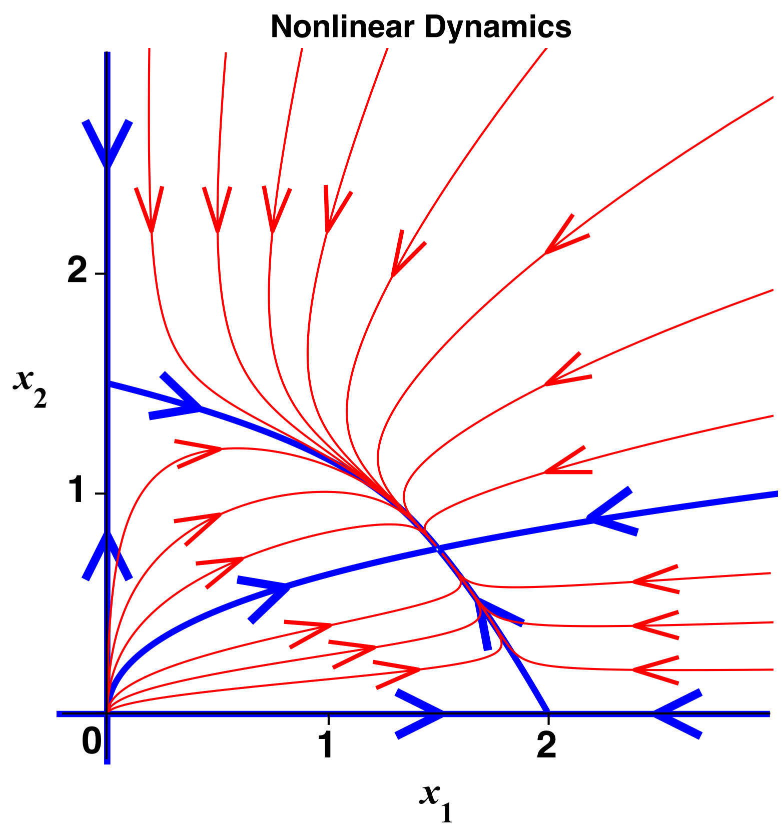
The competition terms

$-2x_2$ and $-x_1$

are "weaker" than

the resource inhibition terms

$-3x_1$ and $-2x_2$



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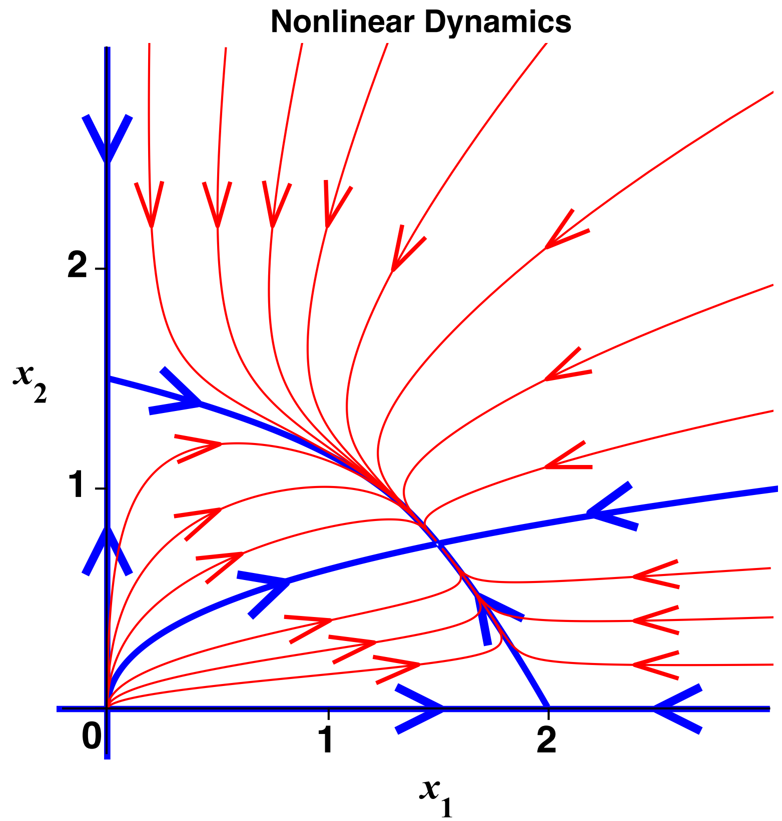

$\det \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} > 0 \quad \Rightarrow$ Weak competition $\quad \Rightarrow$ Stable co-existence

Example 4. Strong Competition Model.

Competing Species:

$$\begin{cases} x_1' = x_1(3 - x_1 - 2x_2) \\ x_2' = x_2(2 - x_1 - x_2) \end{cases}$$

x_2 reduces the growth of x_1

x_1 reduces the growth of x_2

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$$\begin{cases} x_1' = x_1(3 - x_1 - 2x_2) \\ x_2' = x_2(2 - x_1 - x_2) \end{cases}$$

x_2 reduces the growth of x_1

x_1 reduces the growth of x_2

Equilibria:

$$\begin{cases} x_1(3 - x_1 - 2x_2) = 0 \\ x_2(2 - x_1 - x_2) = 0 \end{cases} \implies \text{Separate to four combinations}$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \qquad \begin{cases} 3 - x_1 - 2x_2 = 0 \\ x_2 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ 2 - x_1 - x_2 = 0 \end{cases} \qquad \begin{cases} 3 - x_1 - 2x_2 = 0 \\ 2 - x_1 - x_2 = 0 \end{cases}$$

Four equilibria: $(x_1, x_2) = (0,0), \quad (3,0), \quad (0,2), \quad (1,1).$

Example 4. Linear dynamics near $(0, 0)$

The Linear Approximating System near equilibrium $(0, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Eigenvalues & Eigenvectors:

$$\lambda_1 = 3, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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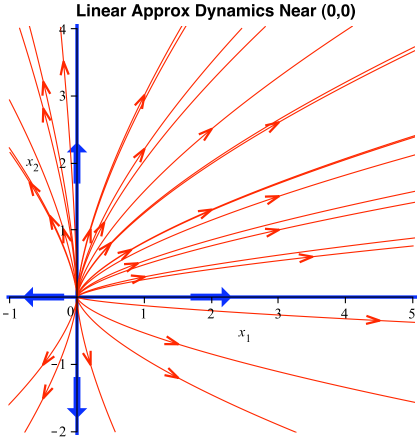

Equilibrium $(0,0)$ is a nodal source.

Example 4. Linear dynamics near $(3, 0)$

The Linear Approximating System near equilibrium $(3, 0)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 3 \\ x_2 \end{bmatrix}$$

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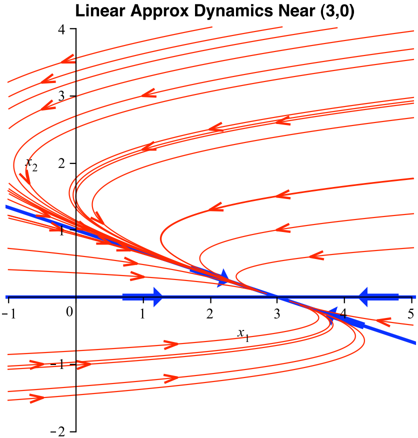

Equilibrium $(3, 0)$ is a nodal sink

Example 4. Linear dynamics near $(0, 2)$

The Linear Approximating System near equilibrium $(0, 2)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

Example 4. Linear dynamics near $(0, 2)$

The Linear Approximating System near equilibrium $(0, 2)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = -2, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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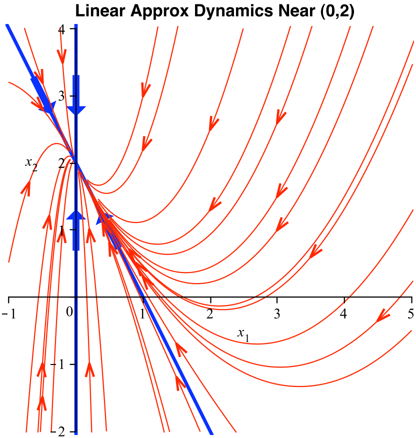

Equilibrium $(0, 2)$ is a nodal sink

Example 4. Linear dynamics near $(1, 1)$

The Linear Approximating System near equilibrium $(1, 1)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

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$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\lambda_1 = -1 + \sqrt{2} > 0$$

$$\vec{\mathbf{w}}_1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 - \sqrt{2} < 0$$

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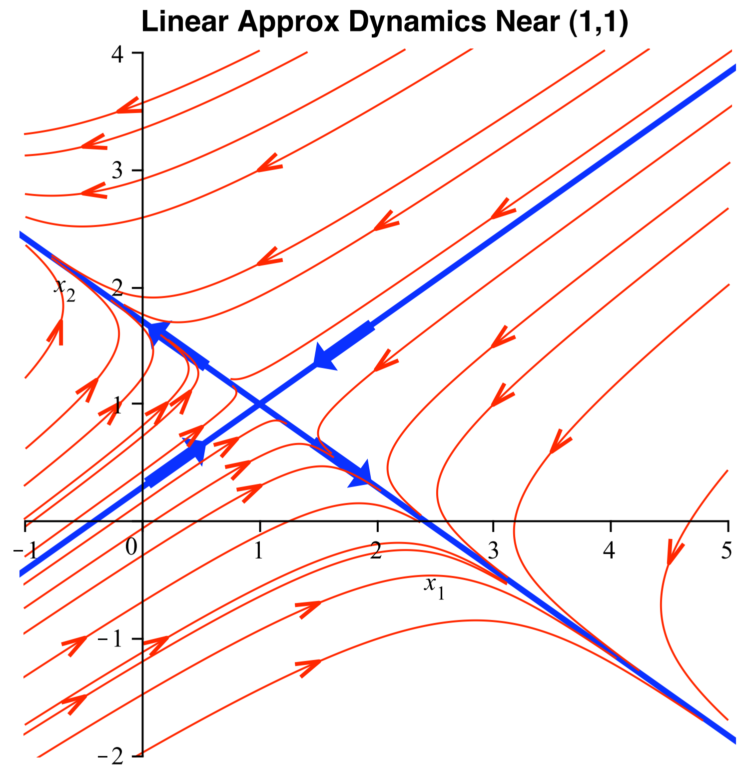
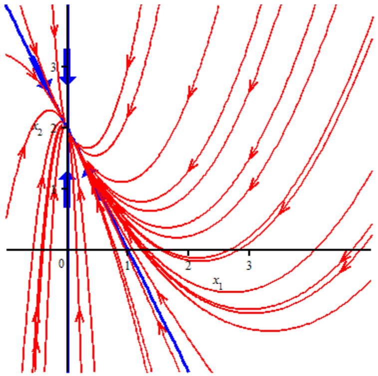
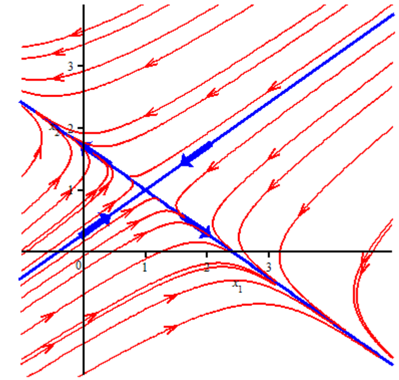


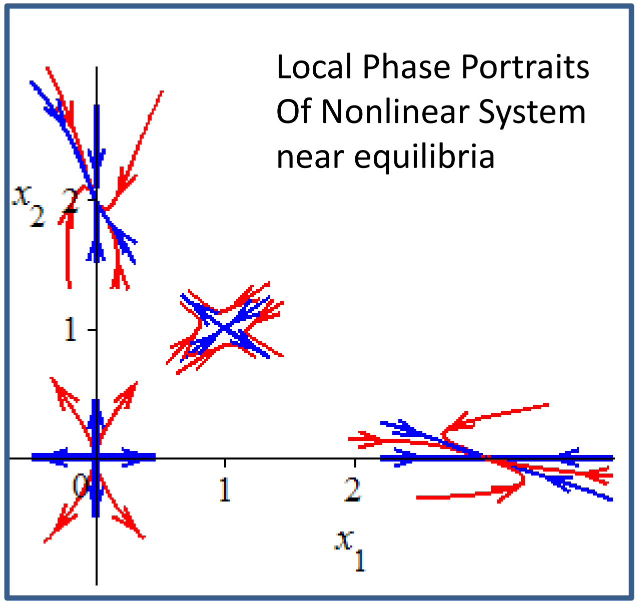
Equilibrium $(1, 1)$ is a saddle

Linear Approx Dynamics Near (0,2)

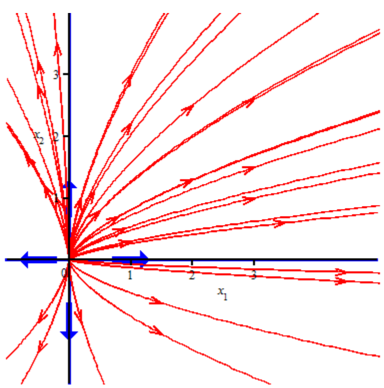


Linear Approx Dynamics Near (1,1)

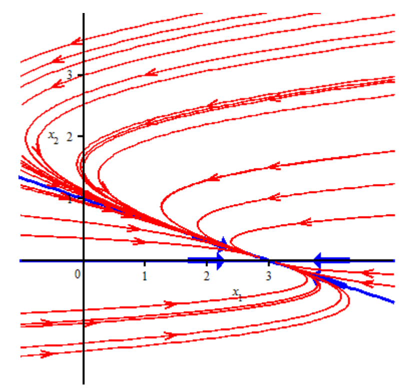




Linear Approx Dynamics Near (0,0)



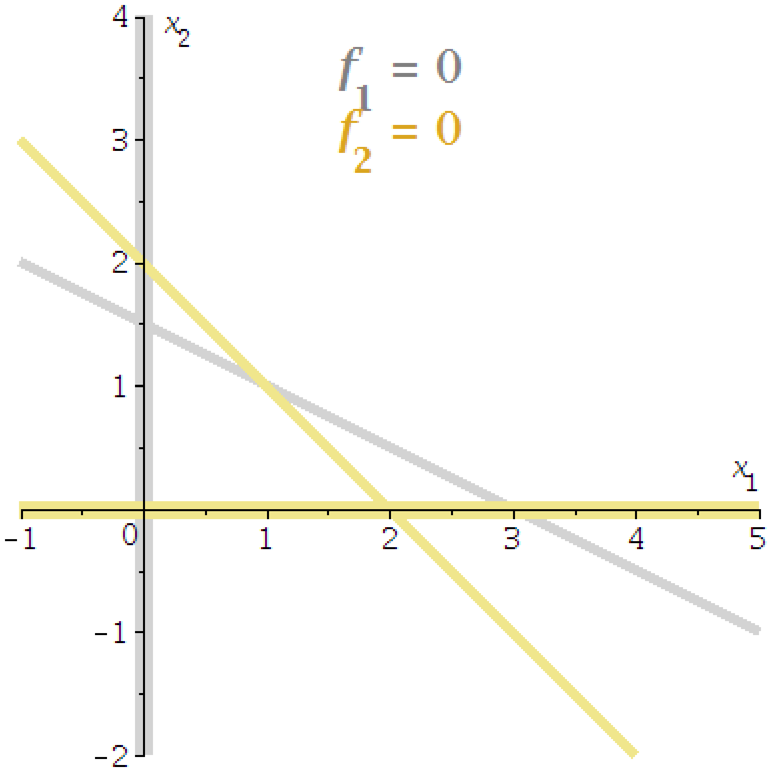
Linear Approx Dynamics Near (3,0)



$$\frac{dx_1}{dt} = x_1(3 - x_1 - 2x_2)$$

$$\frac{dx_2}{dt} = x_2(2 - x_1 - x_2)$$

Nullclines



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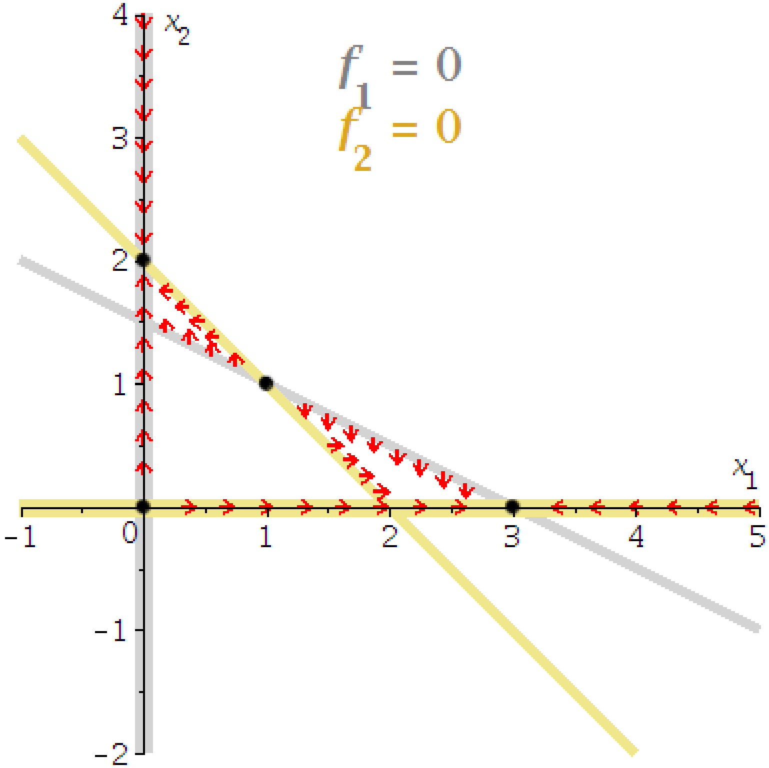
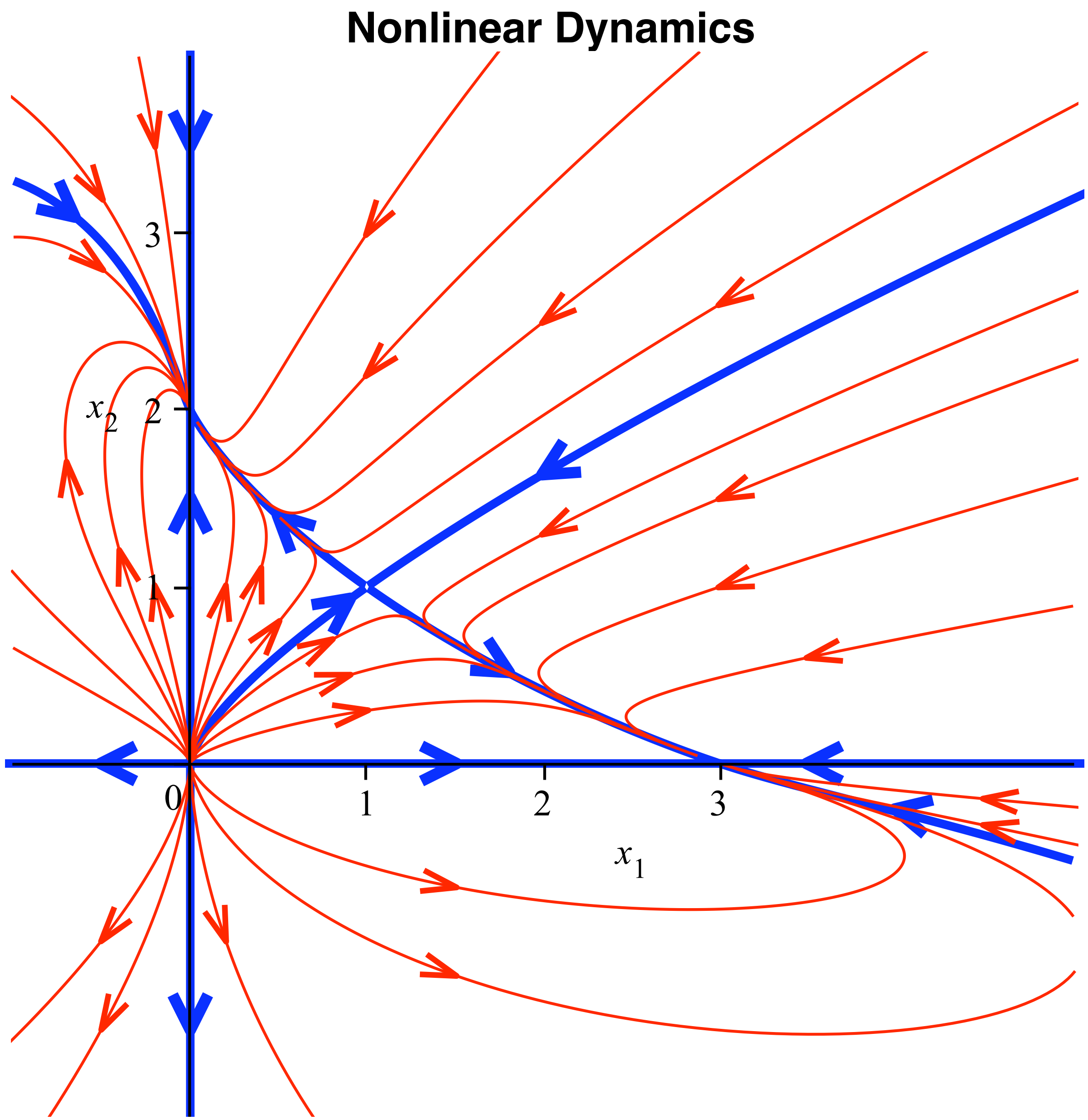
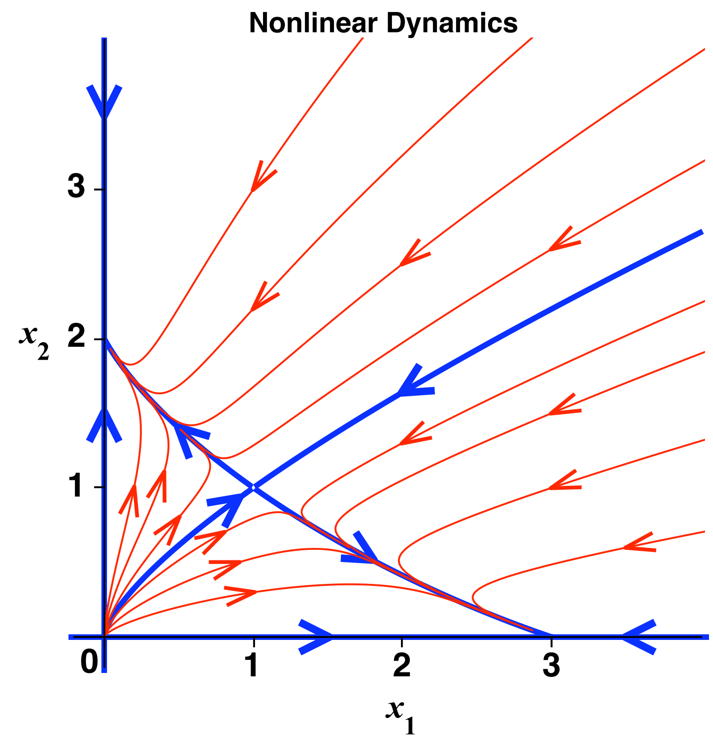
Direction Fields on the Nullclines



Example 4. Global phase portrait

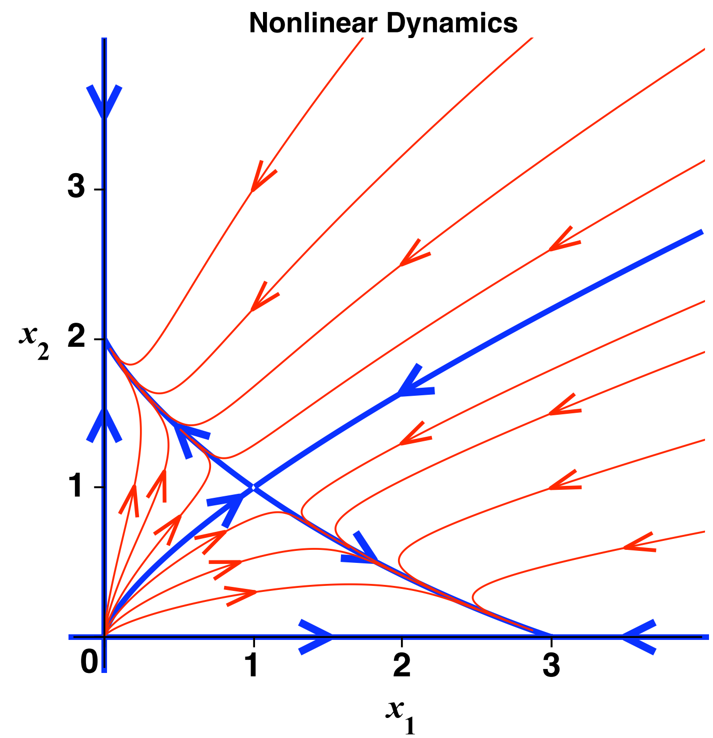


Example 4. Discussion



Example 4. Discussion

- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.



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- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.
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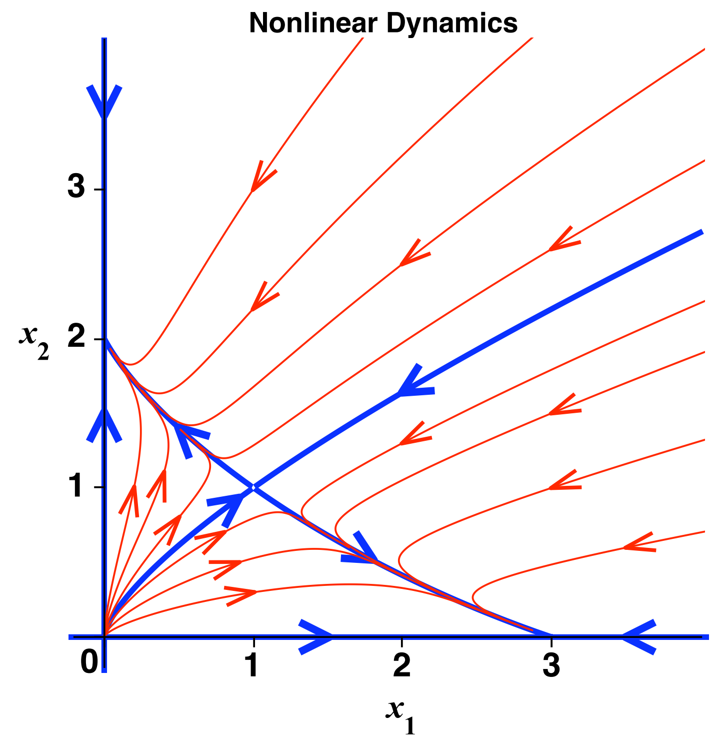
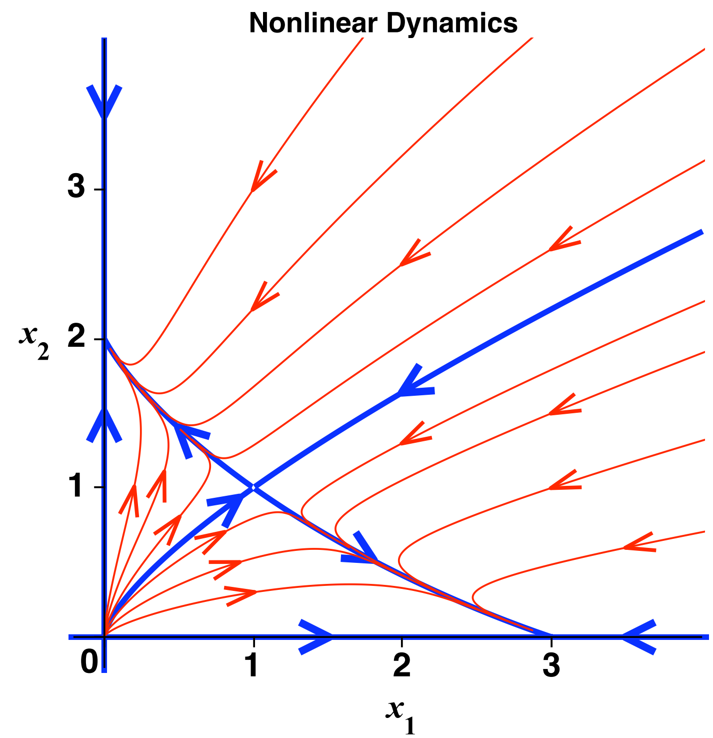


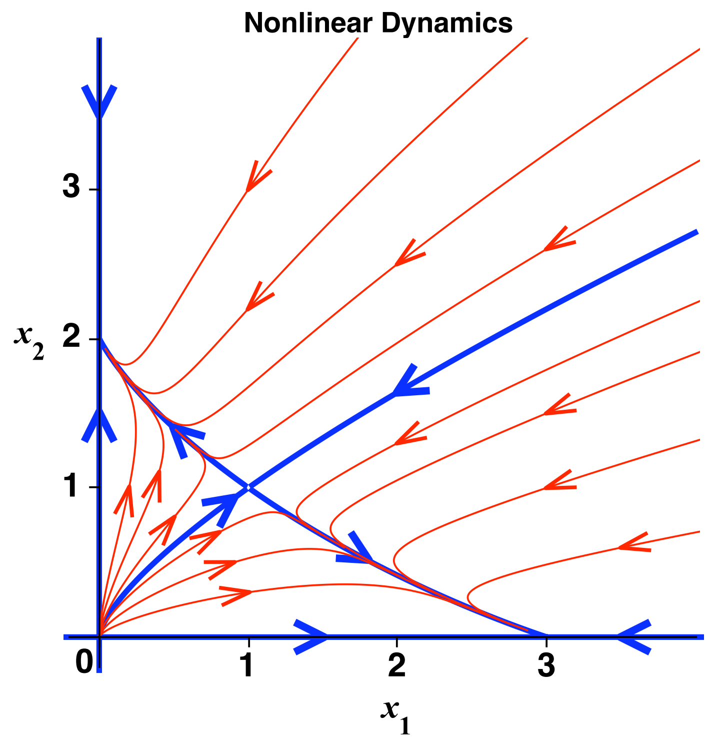
Example 4. Discussion

- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.
- The co-existence state $(1, 1)$ is unstable.
- Almost all positive solutions converge to either $(3, 0)$ or $(0, 2)$.



Example 4. Discussion

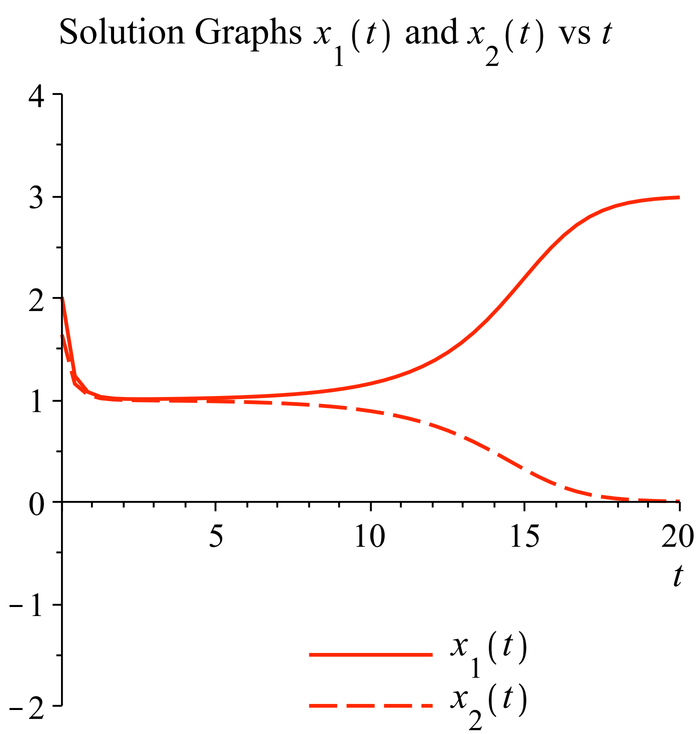
- The survival-extinction states $(3, 0)$ and $(0, 2)$ are both asymptotically stable.
- The co-existence state $(1, 1)$ is unstable.
- Almost all positive solutions converge to either $(3, 0)$ or $(0, 2)$.
- A small difference in the initial conditions may make a huge difference in a species' destiny.



Example 4. (continued. Strong competition)

A small difference in the initial conditions may make a huge difference in a species' destiny.

Initial data:
$x_1(0) = 2.01, x_2(0) = 1.64.$
As $t \to \infty$, $(x_1, x_2) \to (3, 0)$.



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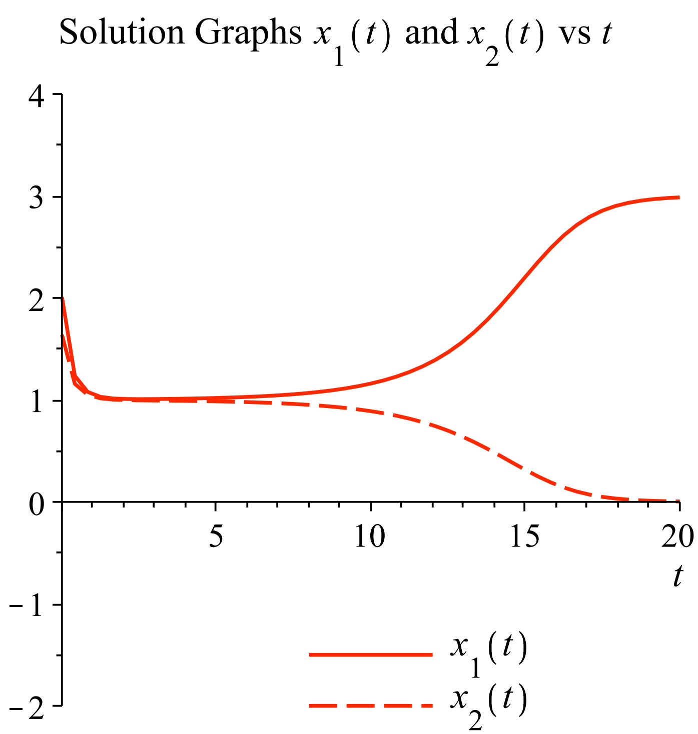

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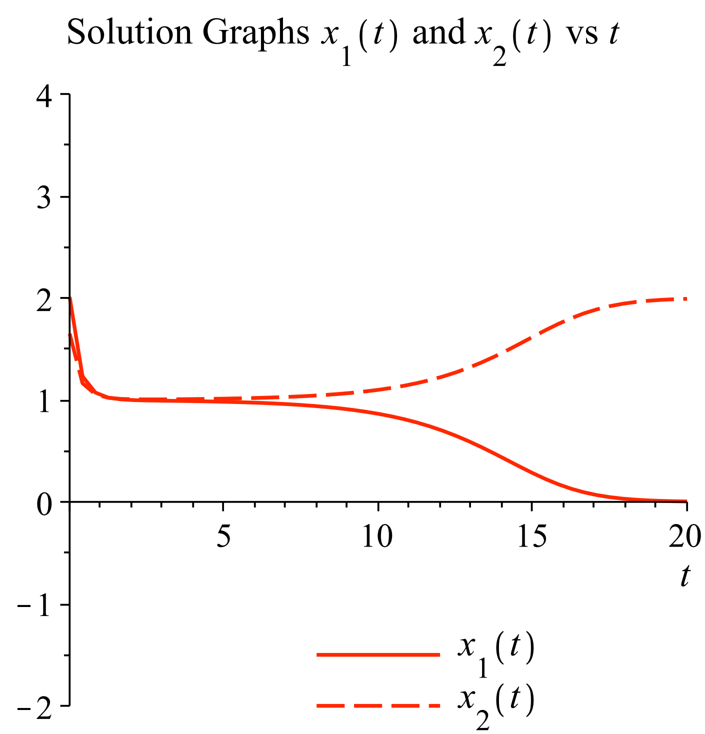
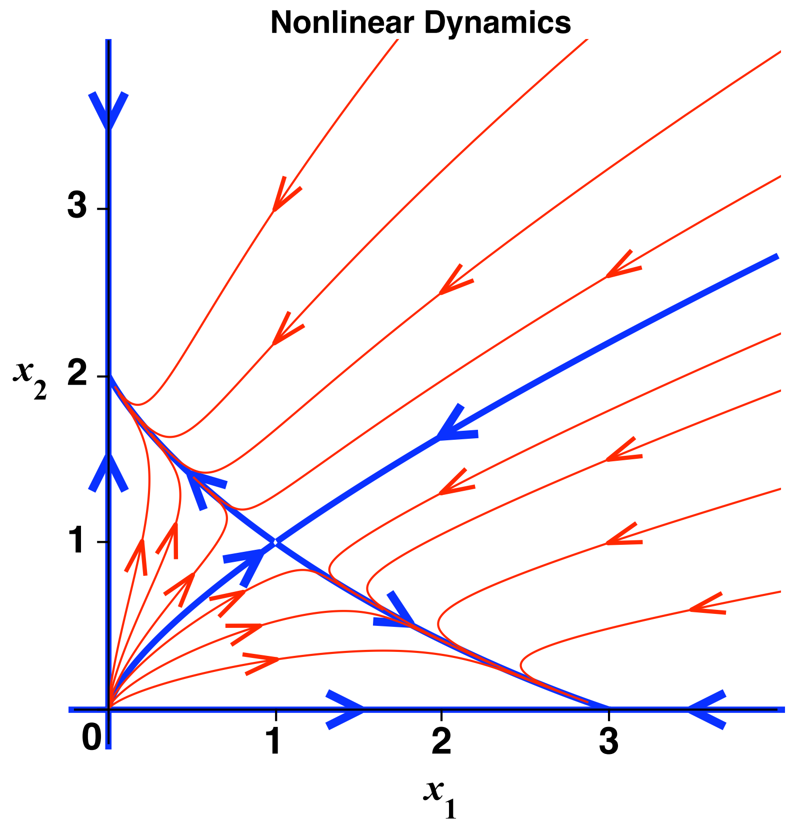


Example 4. (continued. Strong competition)

Question: Why is the co-existence unstable in this system?



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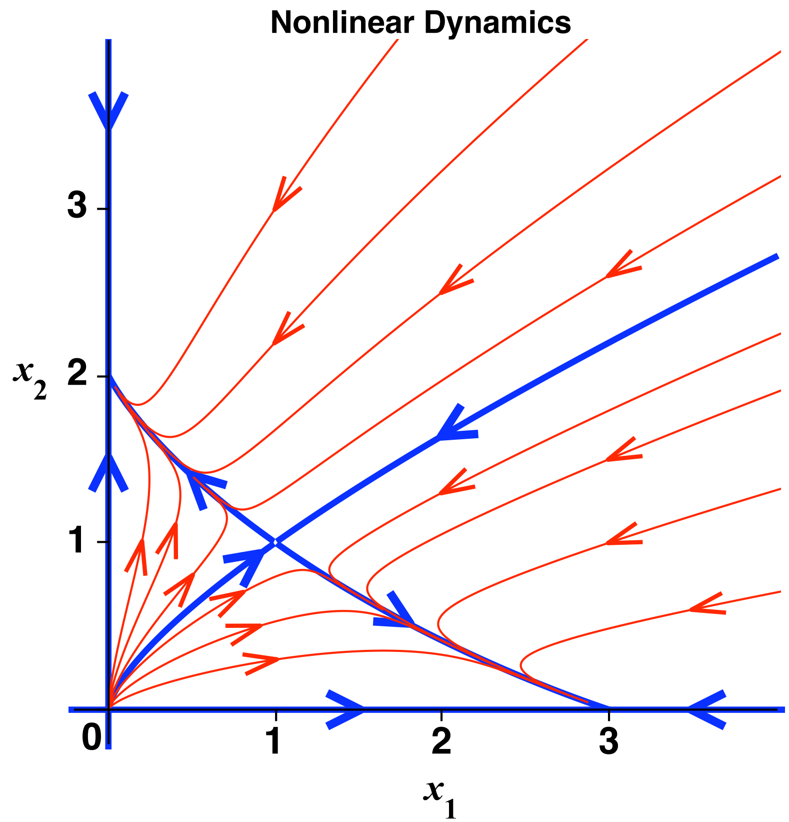
Answer: Strong competition.



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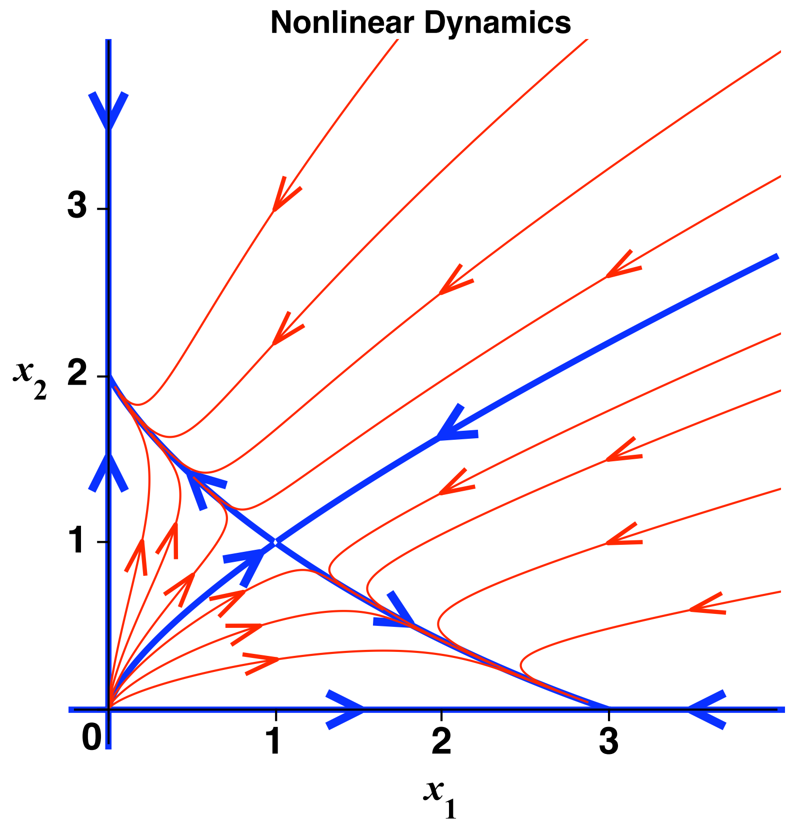


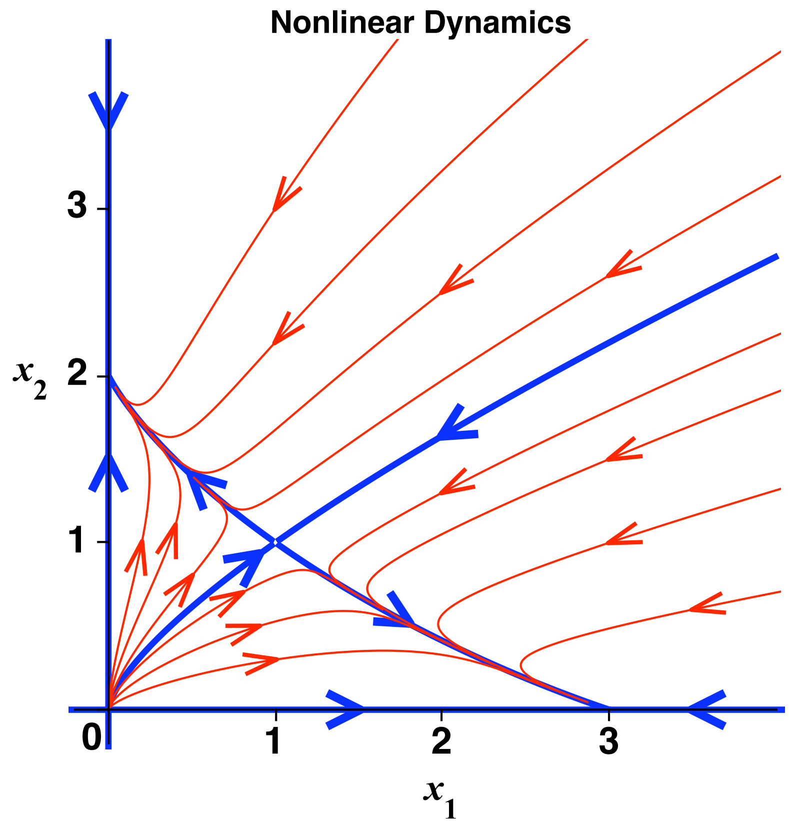
Example 4. (continued. Strong competition)

Question: Why is the co-existence unstable in this system?

Answer: Strong competition.

$$\begin{cases} x_1' = x_1(\ 3\ \ -x_1\ \ -2x_2\ ) \\ x_2' = x_2(\ 2\ \ -x_1\ \ -x_2\ ) \end{cases}$$

The competition terms
$-2x_2$ and $-x_1$



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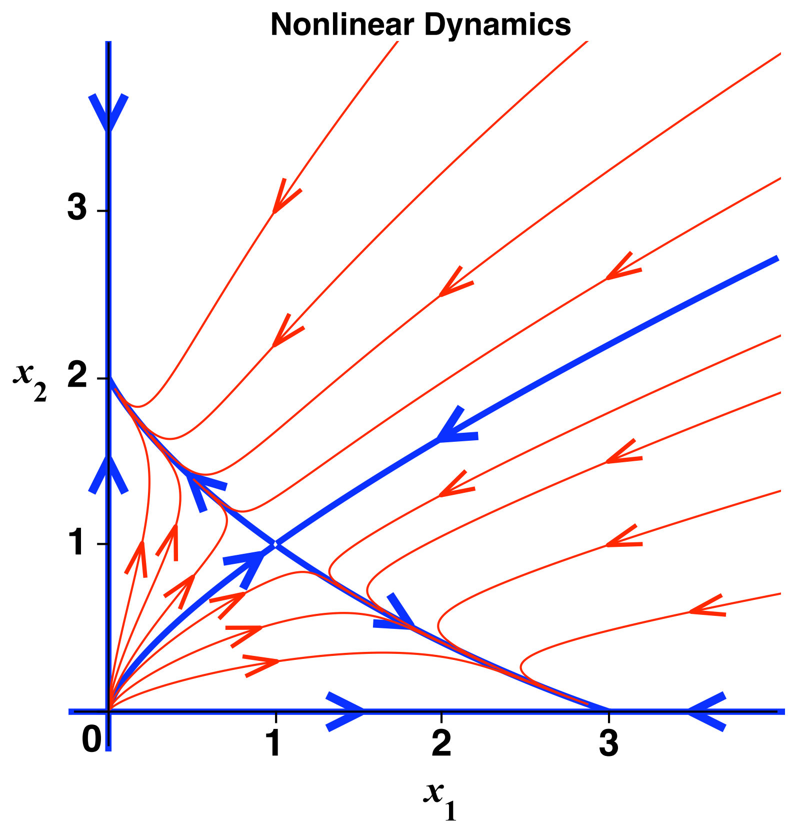

The competition terms

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ARE "STRONGER" THAN

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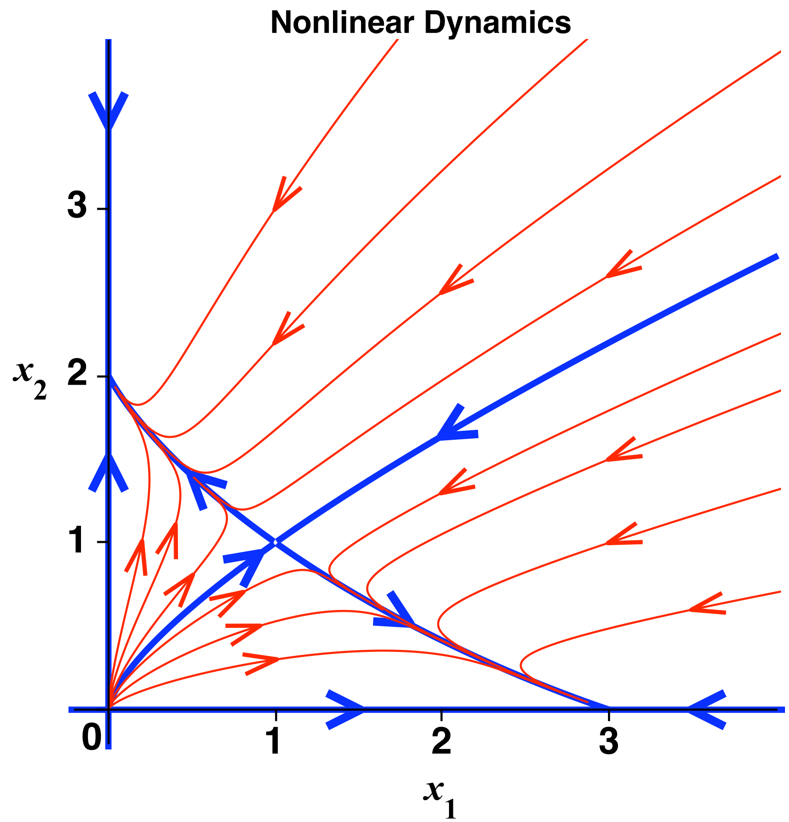

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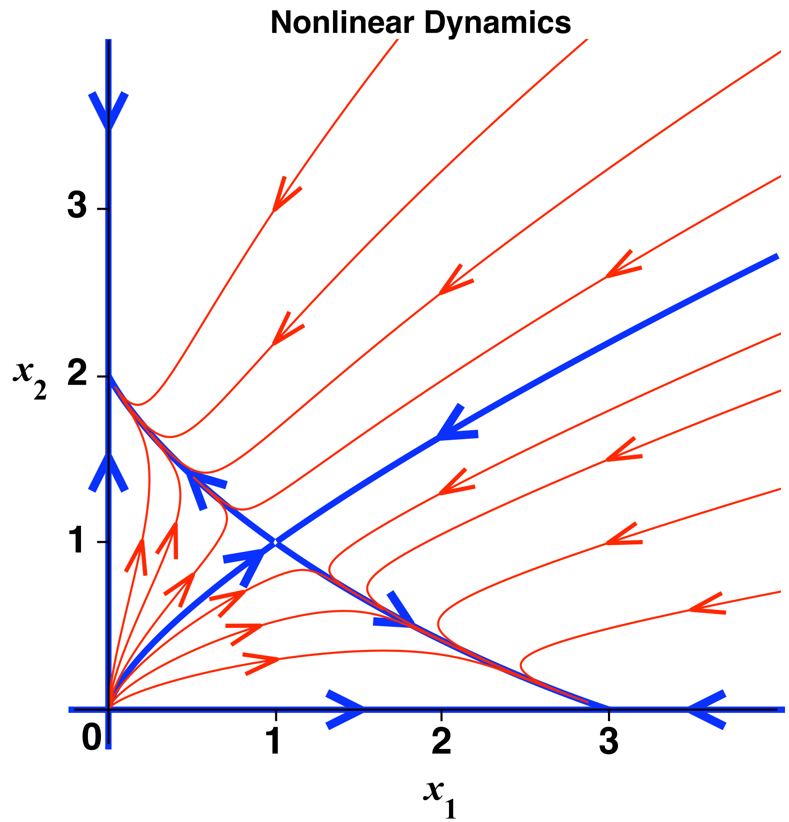

$$\det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} < 0 \ \Rightarrow \text{Strong competition} \ \Rightarrow \begin{cases} \text{One species survives,} \\ \text{the other extincts.} \end{cases}$$