2D Homogeneous Linear Systems with Constant Coefficients — distinct nonzero real eigenvalues

Xu-Yan Chen

Systems of Diff Eqs: $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

where $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, A is a 2×2 real constant matrix

Things to explore:

- General solutions
- Initial value problems
- Geometric figures
  - Solutions graphs x_1 vs t & x_2 vs t
  - Direction fields in the (x_1, x_2) plane
  - Phase portraits in the (x_1, x_2) plane
- Stability/instability of equilibrium $(x_1, x_2) = (0, 0)$

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- If initial condition is given,

  $$\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0 \Rightarrow C_1, C_2 \Rightarrow \text{a unique solution.}$$

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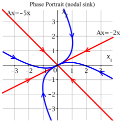
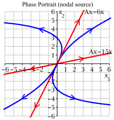
Phase portraits & stability of the equilibrium $(0, 0)$:

Let λ_1 and λ_2 be the eigenvalues of A.

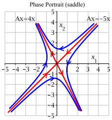
- $\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$
  $\Rightarrow$ Nodal sink,
  asymtotically stable



- $\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2$
  $\Rightarrow$ Nodal source,
  unstable



- $(\lambda_1 > 0, \lambda_2 < 0)$,
  or $(\lambda_1 < 0, \lambda_2 > 0)$
  $\Rightarrow$ Saddle,
  unstable



Example 1.

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

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  $$\det \begin{bmatrix} -3 - \lambda & 2 \\ 1 & -4 - \lambda \end{bmatrix}$$

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$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Partial solutions: } \vec{\mathbf{x}}(t) = Ce^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Need more to get complete solution formula.

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$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow x_1 + x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ An eigenvector $\vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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- General solutions are

$$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where C_1, C_2 are free parameters.

Example 1 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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- The solution to the initial value problem:

$$\vec{\mathbf{x}}(t) = \tfrac{5}{3} e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \tfrac{4}{3} e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

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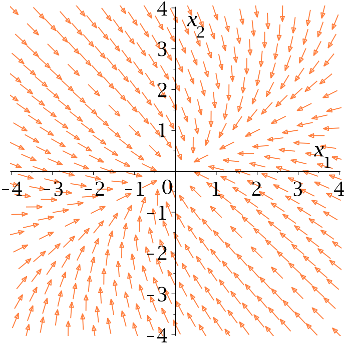

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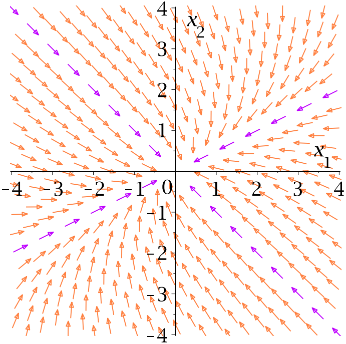

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- $C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ decays to the origin, along the eigenspace of $\lambda_1 = -2$.
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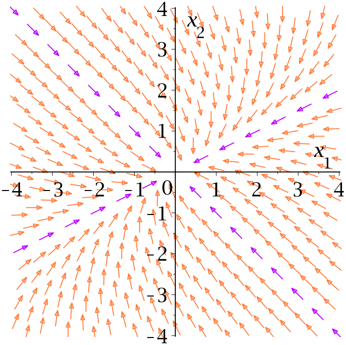

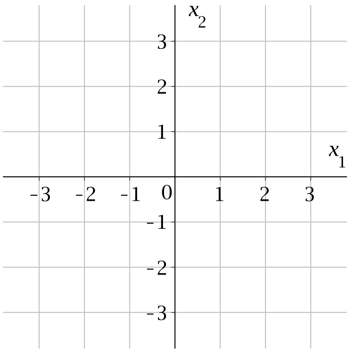

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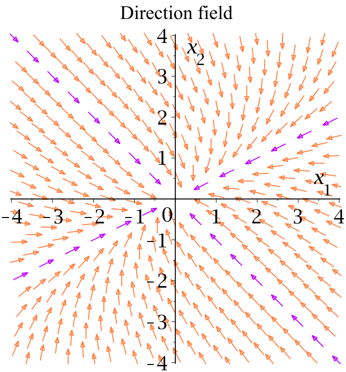

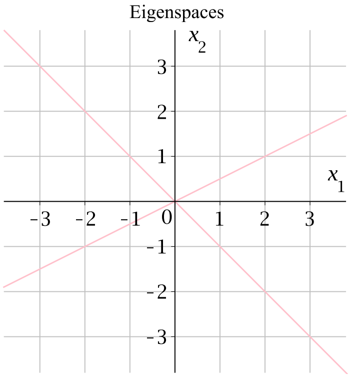

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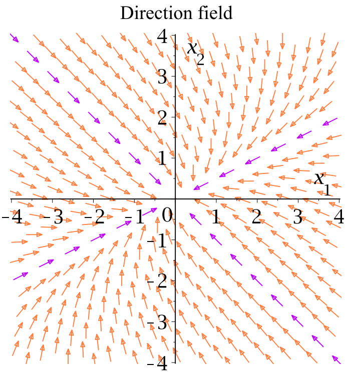

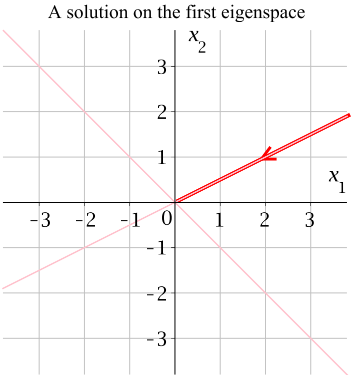

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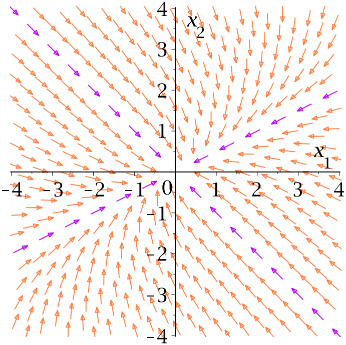
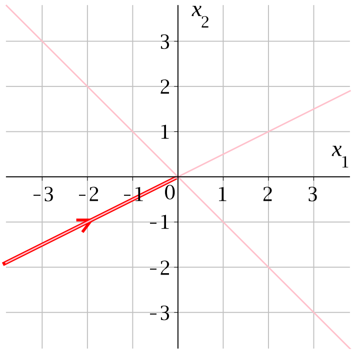


Another solution on the first eigenspace



Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

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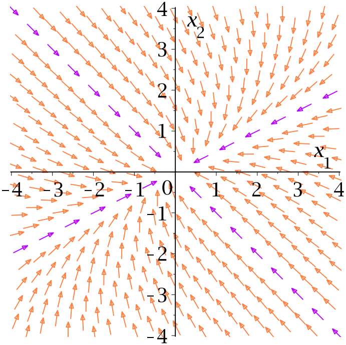
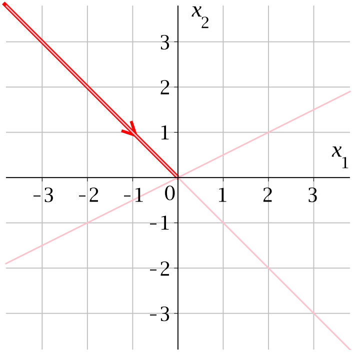
- $C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ decays to the origin, along the eigenspace of $\lambda_1 = -2$.
- $C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ decays to the origin, along the eigenspace of $\lambda_2 = -5$.

Direction field



A solution on the second eigenspace



Example 1 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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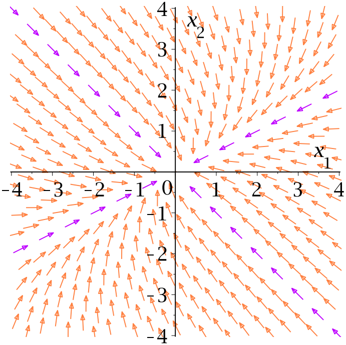
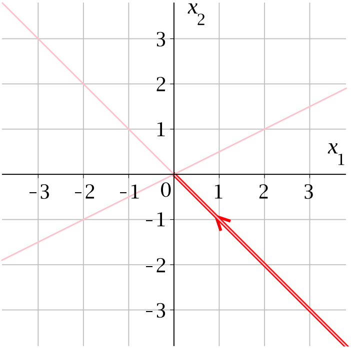
Direction field



Another solution on the second eigenspace



Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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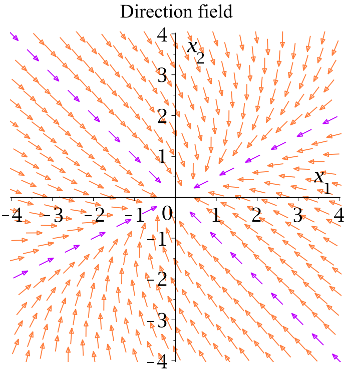
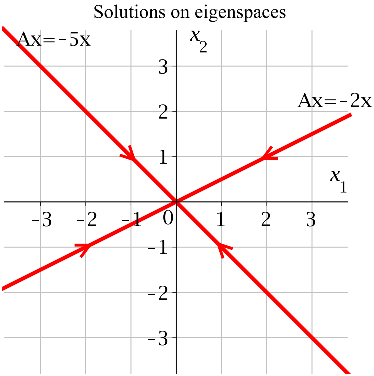


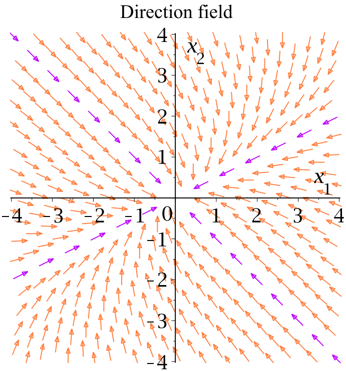


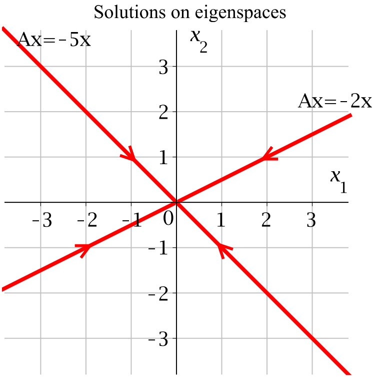
Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When C_1 and C_2 are both $\neq 0$, $\vec{\mathbf{x}}(t) \approx C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as $t \approx \infty$.

It approaches the origin, along the eigenspace of $\lambda_1 = -2$.





Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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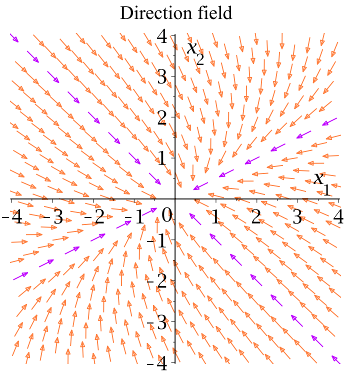

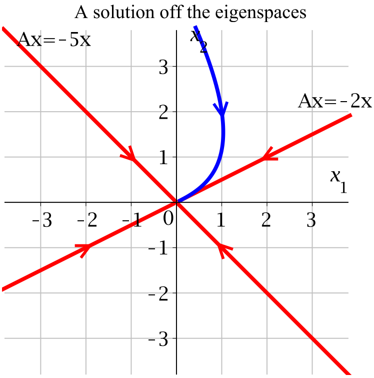

Example 1 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When C_1 and C_2 are both $\neq 0$, $\vec{\mathbf{x}}(t) \approx C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as $t \approx \infty$.

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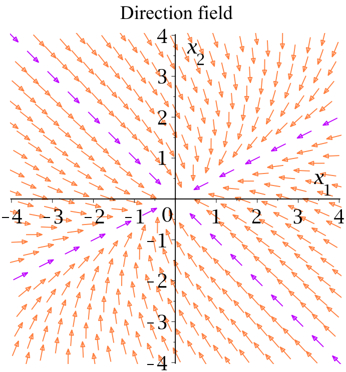

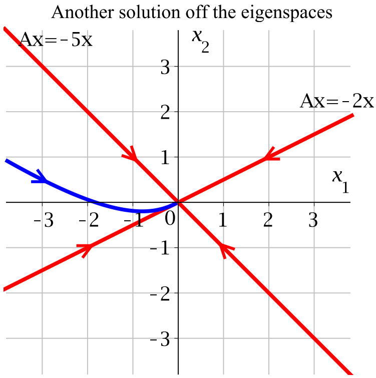

Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

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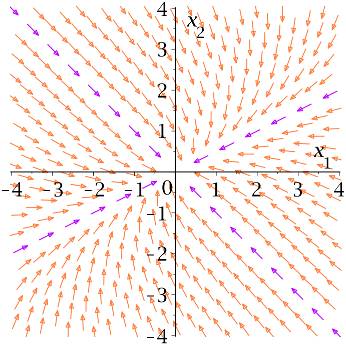
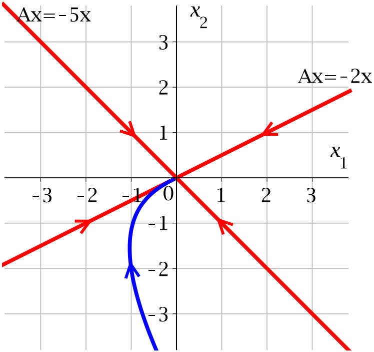
It approaches the origin, along the eigenspace of $\lambda_1 = -2$.

Direction field



Third solution off the eigenspaces



Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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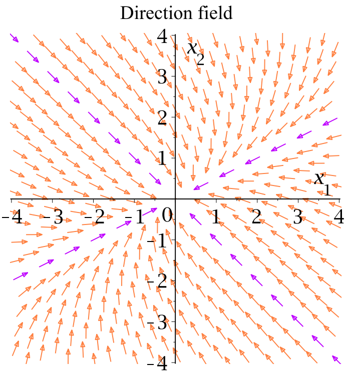

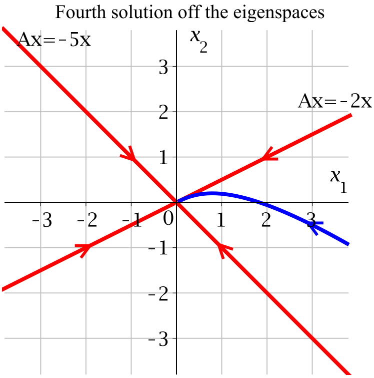

Example 1 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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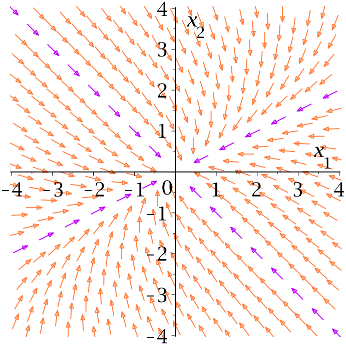
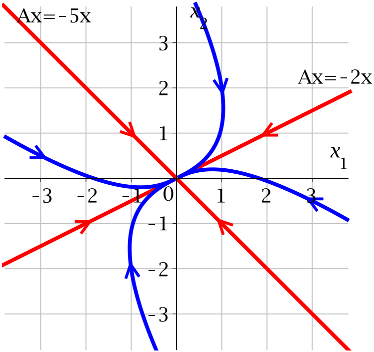


Phase Portrait (nodal sink)



Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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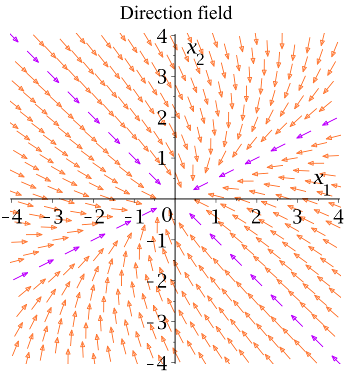

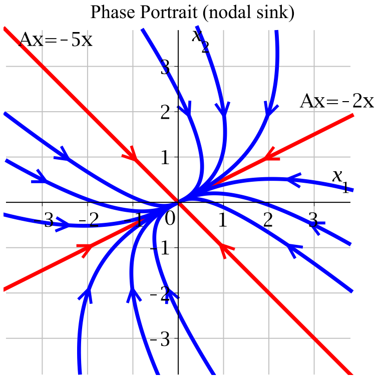

Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{\mathbf{x}}$

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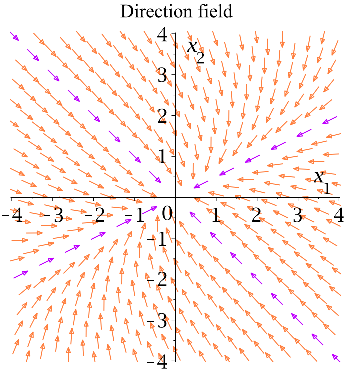
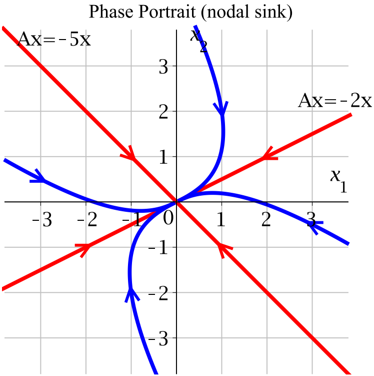
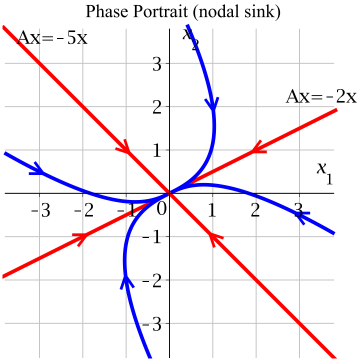




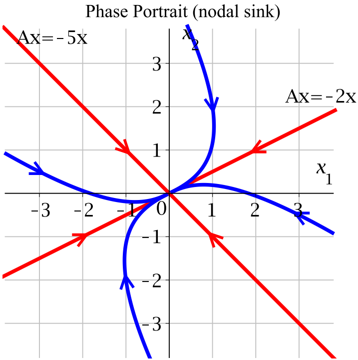
Example 1 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Example 1 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

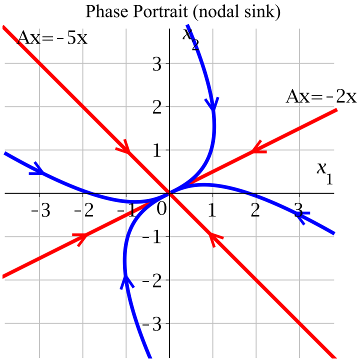
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t}\begin{bmatrix}2\\1\end{bmatrix} + C_2 e^{-5t}\begin{bmatrix}-1\\1\end{bmatrix}$



The equilibrium $(0,0)$ is asymptotically stable.

Example 1 (d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



The equilibrium $(0, 0)$ is asymptotically stable.

We have a *nodal sink*, when $\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$.

Example 2.

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

  $$\det \begin{bmatrix} 16 - \lambda & -5 \\ 2 & 5 - \lambda \end{bmatrix}$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 16 - \lambda & -5 \\ 2 & 5 - \lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 16 - \lambda & -5 \\ 2 & 5 - \lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

$$\Rightarrow \lambda_1 = 15, \lambda_2 = 6$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 16 - \lambda & -5 \\ 2 & 5 - \lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

$$\Rightarrow \lambda_1 = 15, \lambda_2 = 6$$

- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

  $$\det \begin{bmatrix} 16-\lambda & -5 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

  $$\Rightarrow \lambda_1 = 15, \lambda_2 = 6$$

- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

  $$(A - 15I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 16-\lambda & -5 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

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- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A - 15I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow x_1 - 5x_2 = 0$$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

  $$\det \begin{bmatrix} 16 - \lambda & -5 \\ 2 & 5 - \lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

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- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

  $$(A - 15I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  $$\Leftrightarrow x_1 - 5x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 16-\lambda & -5 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

$$\Rightarrow \lambda_1 = 15, \lambda_2 = 6$$

- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A - 15I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow x_1 - 5x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

  $$\det \begin{bmatrix} 16-\lambda & -5 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 21\lambda + 90 = 0$$

  $$\Rightarrow \lambda_1 = 15, \lambda_2 = 6$$

- Eigenvectors of A for $\lambda_1 = 15$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

  $$(A - 15I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  $$\Leftrightarrow x_1 - 5x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

  $$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

  $$\Rightarrow \text{Partial solutions: } \vec{\mathbf{x}}(t) = Ce^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

  Need more to get complete solution formula.

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = 15, \lambda_2 = 6$
- An eigenvector for $\lambda_1 = 15$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = 15, \lambda_2 = 6$
- An eigenvector for $\lambda_1 = 15$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$
- An eigenvector for $\lambda_2 = 6$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 6I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 10 & -5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 2x_1 - x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = 15, \lambda_2 = 6$
- An eigenvector for $\lambda_1 = 15$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$
- An eigenvector for $\lambda_2 = 6$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 6I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 10 & -5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 2x_1 - x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- General solutions are

$$\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where C_1, C_2 are free parameters.

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 13/9 \end{bmatrix}$$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 13/9 \end{bmatrix}$$

- The solution to the initial value problem:

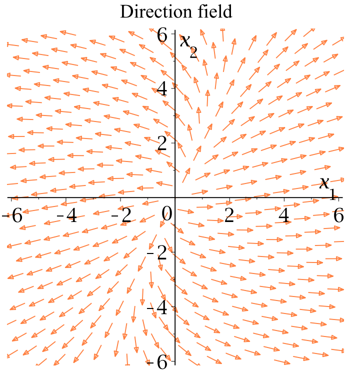
$$\vec{\mathbf{x}}(t) = \tfrac{1}{9} e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \tfrac{13}{9} e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

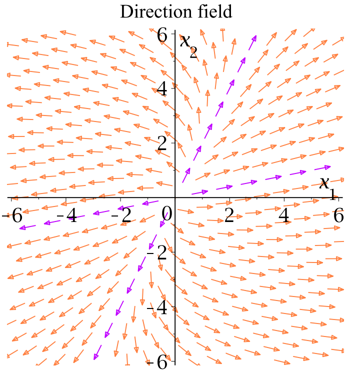
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

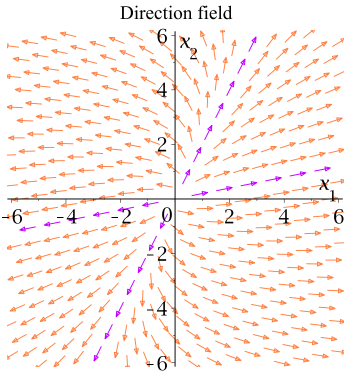
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

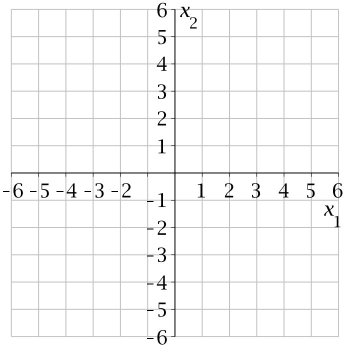


Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

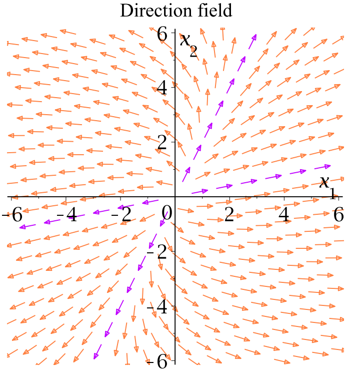
- $C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ leaves away, along the eigenspace of $\lambda_1 = 15$.
- $C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ leaves away, along the eigenspace of $\lambda_2 = 6$.

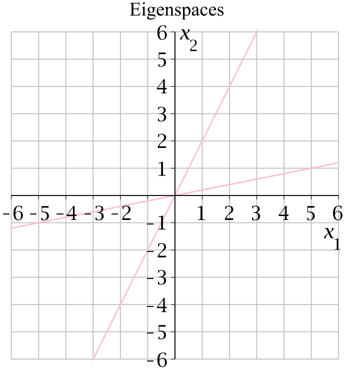




Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

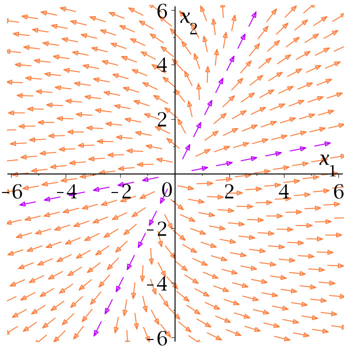




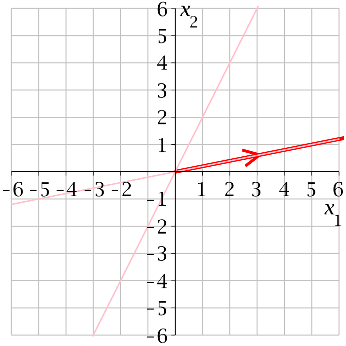
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Direction field



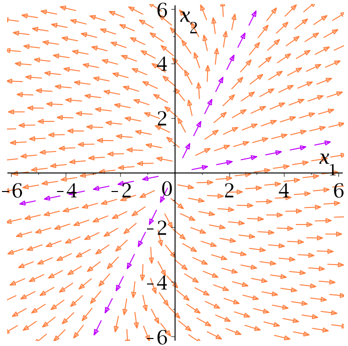
A solution on the first eigenspace



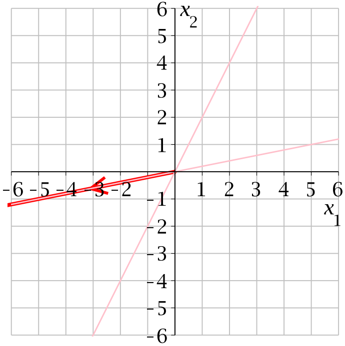
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Direction field

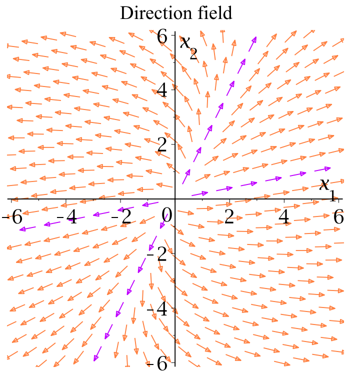


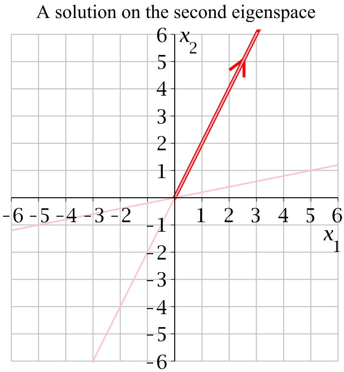
Another solution on the first eigenspace



Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

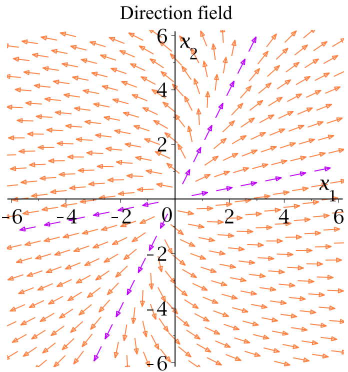
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

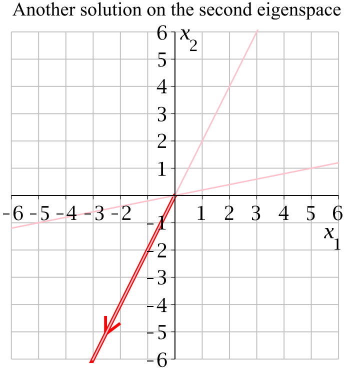




Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

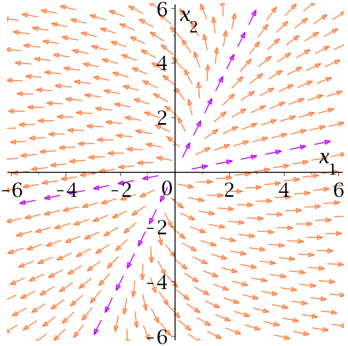




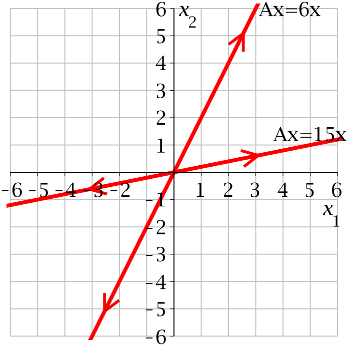
Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Direction field



Solutions on eigenspaces

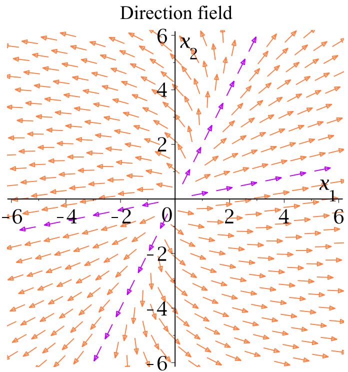


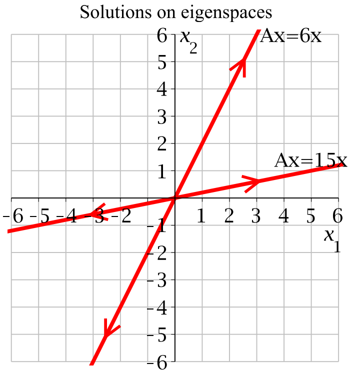
Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

When C_1 and C_2 are both $\neq 0$,

$\vec{\mathbf{x}}(t) \approx C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ as $t \approx \infty$; $\qquad \vec{\mathbf{x}}(t) \approx C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as $t \approx -\infty$.





Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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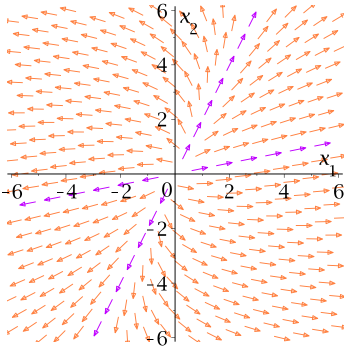
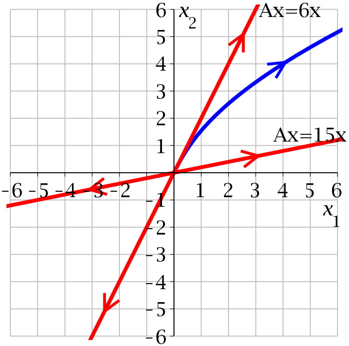
Direction field



A solution off the eigenspaces



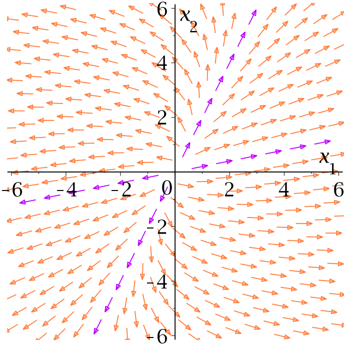
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

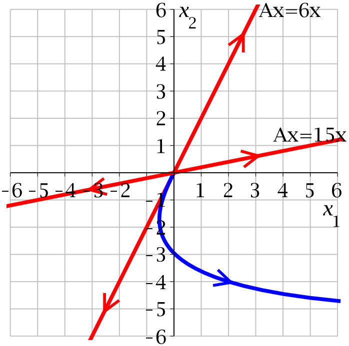
When C_1 and C_2 are both $\neq 0$,

$\vec{\mathbf{x}}(t) \approx C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ as $t \approx \infty$; $\vec{\mathbf{x}}(t) \approx C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as $t \approx -\infty$.

Direction field



Another solution off the eigenspaces



Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

When C_1 and C_2 are both $\neq 0$,

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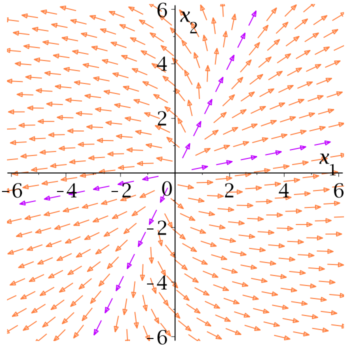
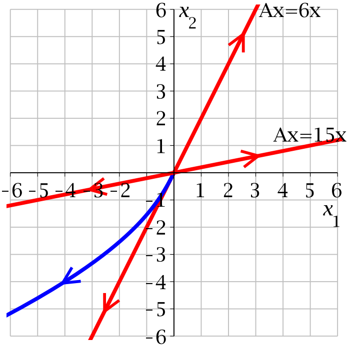
Direction field



Third solution off the eigenspaces



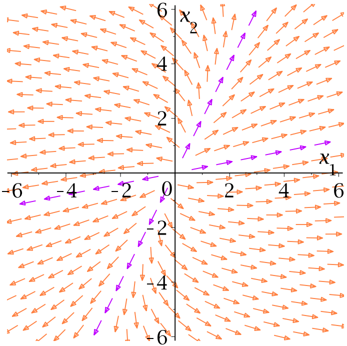
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

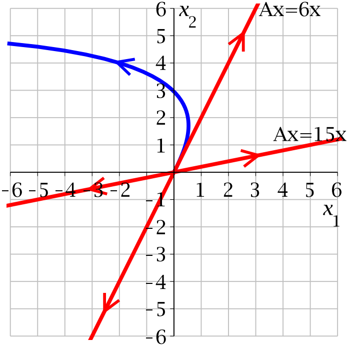
When C_1 and C_2 are both $\neq 0$,

$\vec{\mathbf{x}}(t) \approx C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ as $t \approx \infty$; $\qquad \vec{\mathbf{x}}(t) \approx C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as $t \approx -\infty$.

Direction field



Fourth solution off the eigenspaces



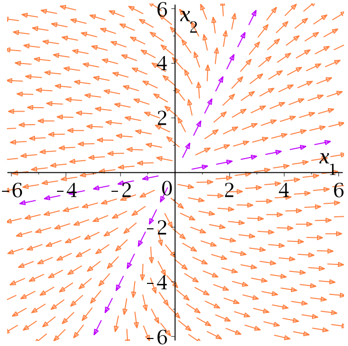
Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 16 & -5 \\ 2 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

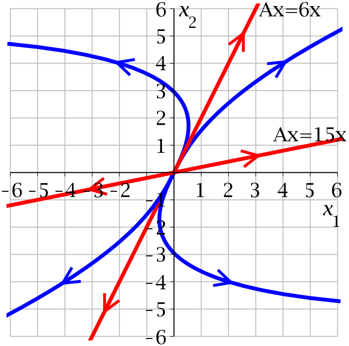
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field

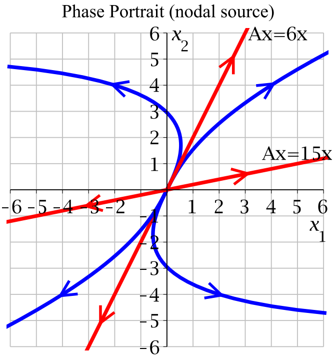


Phase Portrait (nodal source)



Example 2 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



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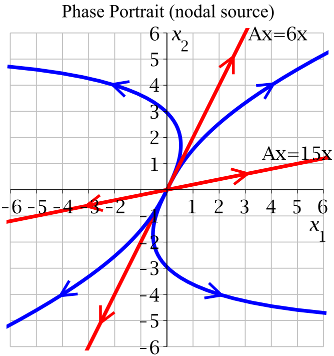
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{15t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



The equilibrium $(0,0)$ is unstable.

Example 2 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

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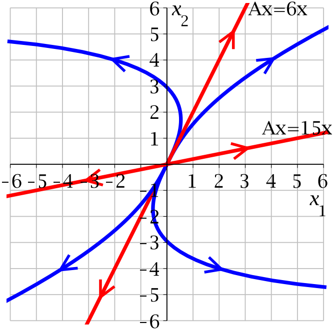
Phase Portrait (nodal source)



The equilibrium $(0,0)$ is unstable.

We have a *nodal source*,
when $\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2$.

Example 3.

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix}$$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

  $$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

  $$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

- Eigenvectors of A for $\lambda_1 = -5$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

- Eigenvectors of A for $\lambda_1 = -5$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

- Eigenvectors of A for $\lambda_1 = -5$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 3x_1 - 3x_2 = 0$$

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

- Eigenvectors of A for $\lambda_1 = -5$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 3x_1 - 3x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -2-\lambda & -3 \\ -6 & 1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 4$$

- Eigenvectors of A for $\lambda_1 = -5$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 3x_1 - 3x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = -5, \lambda_2 = 4$
- An eigenvector for $\lambda_1 = -5$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = -5, \lambda_2 = 4$
- An eigenvector for $\lambda_1 = -5$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- An eigenvector for $\lambda_2 = 4$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 4I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -6 & -3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow -6x_1 - 3x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Example 3 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = -5, \lambda_2 = 4$
- An eigenvector for $\lambda_1 = -5$: $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- An eigenvector for $\lambda_2 = 4$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 4I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -6 & -3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow -6x_1 - 3x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- General solutions are

$$\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

where C_1, C_2 are free parameters.

Example 3 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 3 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Example 3 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}$$

Example 3 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}$$

- The solution to the initial value problem:

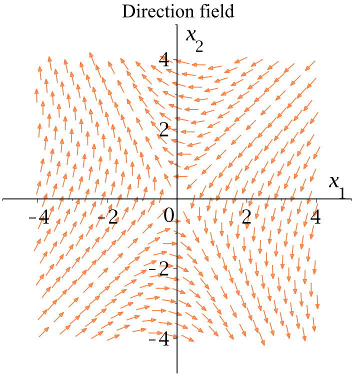
$$\vec{\mathbf{x}}(t) = \tfrac{4}{3} e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{1}{3} e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

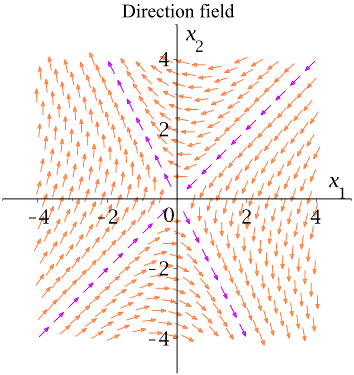
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

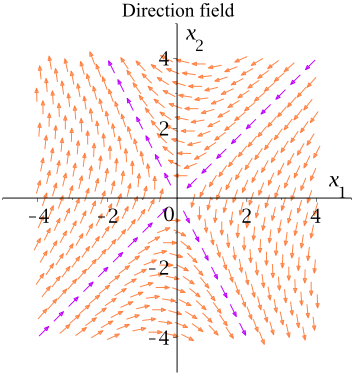
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

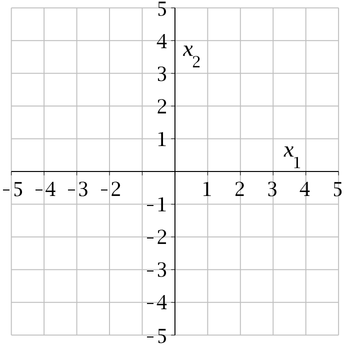


Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

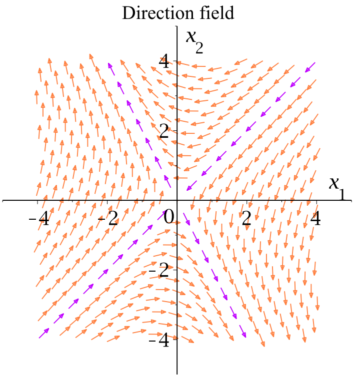
- $C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, along the eigenspace of $\lambda_1 = -5$.
- $C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ leaves away, along the eigenspace of $\lambda_2 = 4$.

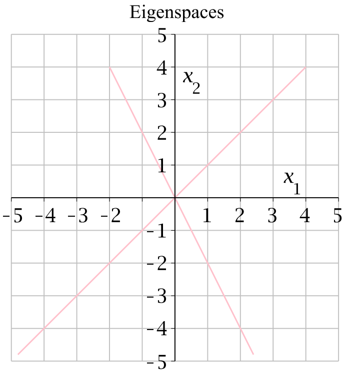




Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

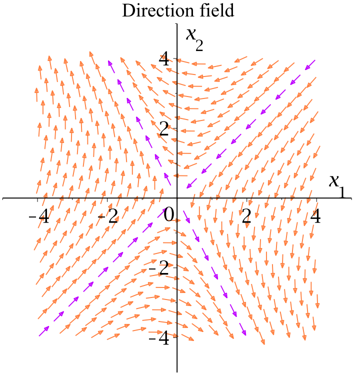
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

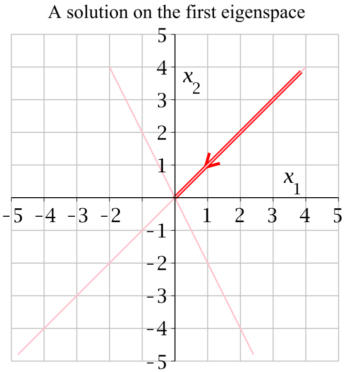




Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

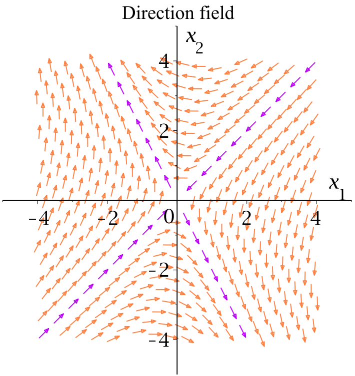
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

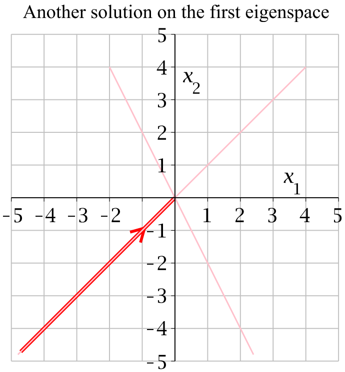




Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

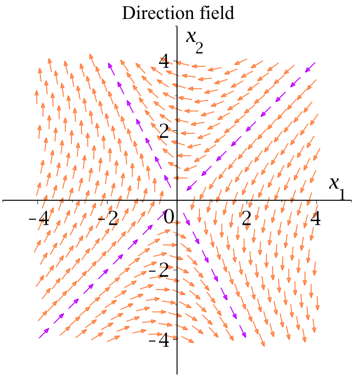
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

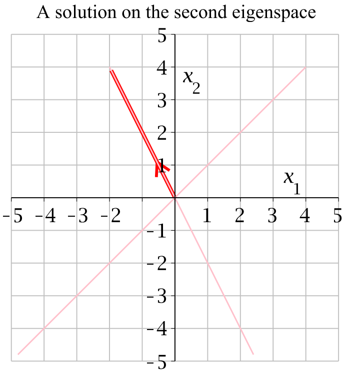




Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

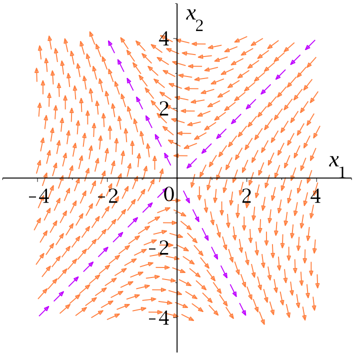




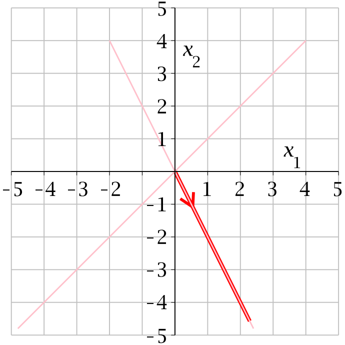
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Direction field

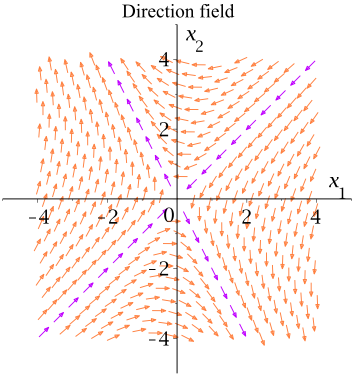


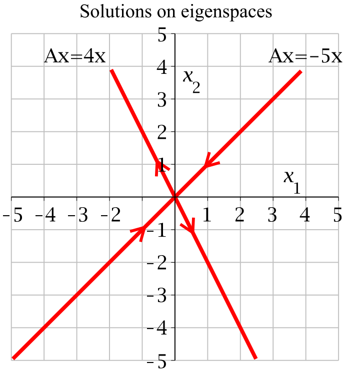
Another solution on the second eigenspace



Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$





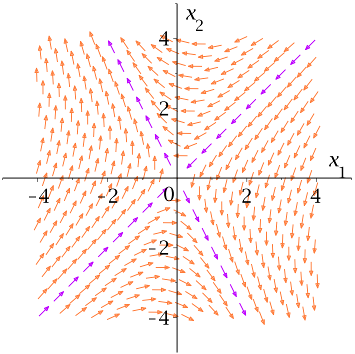
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

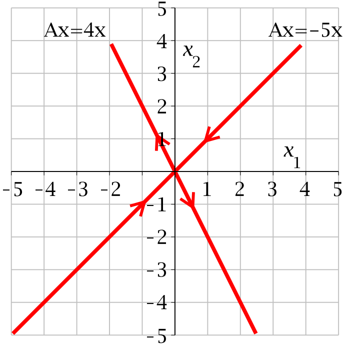
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field



Solutions on eigenspaces

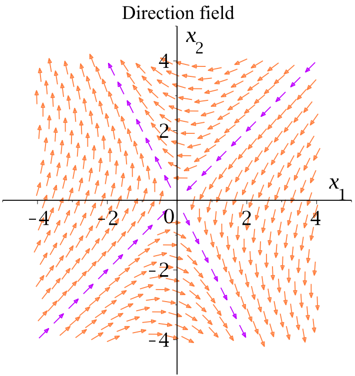


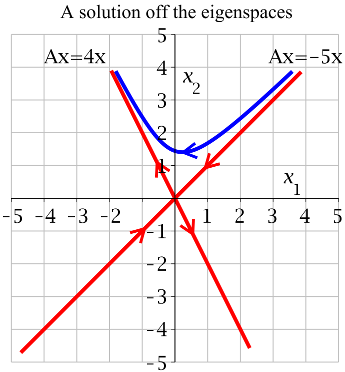
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

When C_1 and C_2 are both $\neq 0$,

$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as $t \approx \infty$; $\vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $t \approx -\infty$.





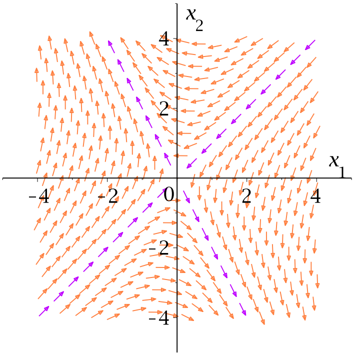
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

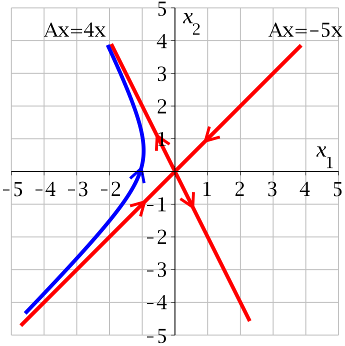
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field



Another solution off the eigenspaces



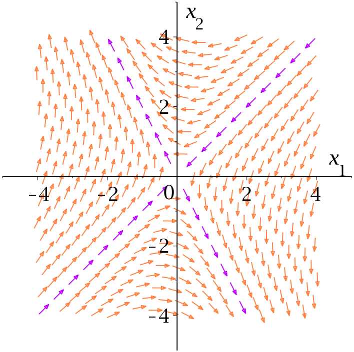
Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

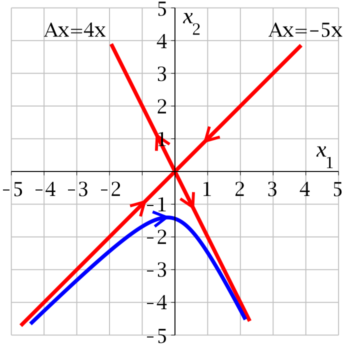
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field



Third solution off the eigenspaces



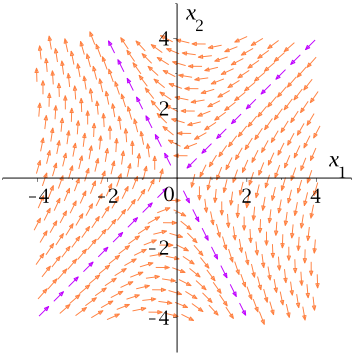
Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

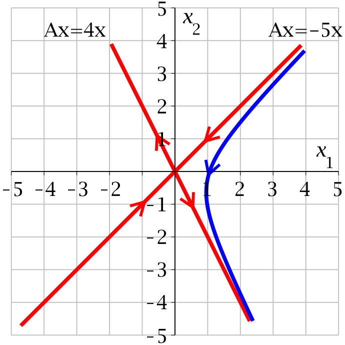
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field



Fourth solution off the eigenspaces



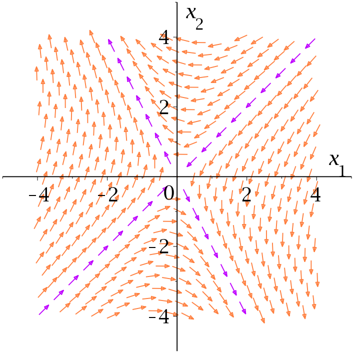
Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -2 & -3 \\ -6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

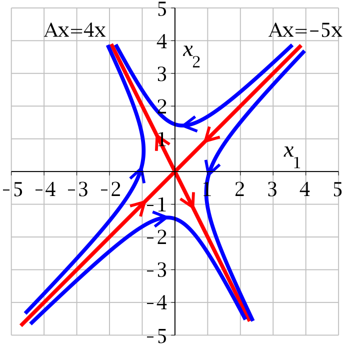
When C_1 and C_2 are both $\neq 0$,

$$\vec{\mathbf{x}}(t) \approx C_2 e^{4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ as } t \approx \infty; \qquad \vec{\mathbf{x}}(t) \approx C_1 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \approx -\infty.$$

Direction field

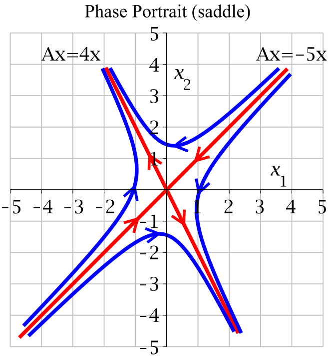


Phase Portrait (saddle)



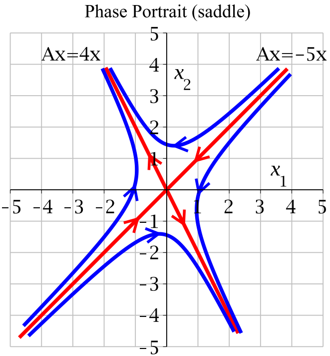
Example 3 (d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

General solutions: $$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Example 3 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Most solutions go to space infinity.

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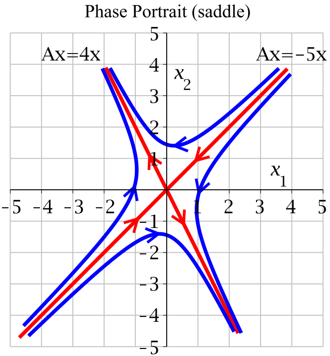

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We have a *saddle*,
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eigenvalues co-exist.

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