2D Homogeneous Linear Systems with Constant Coefficients: a zero eigenvalue $\Rightarrow$ a line of equilibria

Xu-Yan Chen

Systems of Diff Eqs: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

where $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, A is a 2×2 real constant matrix

Things to explore:

- General solutions
- Initial value problems
- Geometric figures
  - Solutions graphs x_1 vs t & x_2 vs t
  - Direction fields in the (x_1, x_2) plane
  - Phase portraits in the (x_1, x_2) plane
- Stability/instability of equilibrium $(x_1, x_2) = (0, 0)$

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

What if we have a zero eigenvalue?
Assume that the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

Solution Method: (nothing new)

- Find an eigenvector $\vec{\mathbf{u}}_1$ for λ_1, by solving
  $$(A - \lambda_1 I)\vec{\mathbf{x}} = 0$$
- Find an eigenvector $\vec{\mathbf{u}}_2$ for λ_2, by solving
  $$(A - \lambda_2 I)\vec{\mathbf{x}} = 0.$$
- General solutions are
  $$\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$$

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

What if we have a zero eigenvalue?
Assume that the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

Solution Method: (nothing new)

- Find an eigenvector $\vec{\mathbf{u}}_1$ for λ_1, by solving
  $$(A - \lambda_1 I)\vec{\mathbf{x}} = 0 \qquad \Rightarrow A\vec{\mathbf{x}} = 0$$
- Find an eigenvector $\vec{\mathbf{u}}_2$ for λ_2, by solving
  $$(A - \lambda_2 I)\vec{\mathbf{x}} = 0.$$
- General solutions are
  $$\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$$

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

What if we have a zero eigenvalue?
Assume that the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

Solution Method: (nothing new)

- Find an eigenvector $\vec{\mathbf{u}}_1$ for λ_1, by solving
  $$(A - \lambda_1 I)\vec{\mathbf{x}} = 0 \qquad \Rightarrow A\vec{\mathbf{x}} = 0$$
- Find an eigenvector $\vec{\mathbf{u}}_2$ for λ_2, by solving
  $$(A - \lambda_2 I)\vec{\mathbf{x}} = 0.$$
- General solutions are
  $$\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2 \qquad \Rightarrow \vec{\mathbf{x}}(t) = C_1\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$$

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

Phase portraits & stability

when the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

General solutions: $\vec{\mathbf{x}}(t) = C_1\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

Phase portraits & stability

when the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

General solutions: $\vec{\mathbf{x}}(t) = C_1\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$

On the eigenspace of the zero eigenvalue λ_1, the solutions are time independent; i.e., the eigenspace of the zero eigenvalue is a line of equilibria.

2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

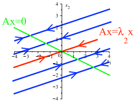
Phase portraits & stability

when the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

General solutions: $\vec{\mathbf{x}}(t) = C_1\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$

On the eigenspace of the zero eigenvalue λ_1, the solutions are time independent; i.e., the eigenspace of the zero eigenvalue is a line of equilibria.

- $\lambda_1 = 0, \lambda_2 < 0$
  $\Rightarrow$ Attractive line of equilibria,
  stable,
  but not asymptotically stable



2D Systems: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

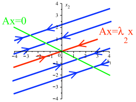
Phase portraits & stability

when the eigenvalues of A are: $\lambda_1 = 0, \lambda_2 \neq 0$.

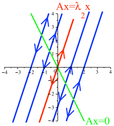
General solutions: $\vec{\mathbf{x}}(t) = C_1\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2$

On the eigenspace of the zero eigenvalue λ_1, the solutions are time independent; i.e., the eigenspace of the zero eigenvalue is a line of equilibria.

- $\lambda_1 = 0, \lambda_2 < 0$
  $\Rightarrow$ Attractive line of equilibria,
  stable,
  but not asymptotically stable



- $\lambda_1 = 0, \lambda_2 > 0$
  $\Rightarrow$ Repulsive line of equilibria,
  unstable



Example 1 (an attractive line of equilibria).

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 1 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

Example 1 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -3-\lambda & -6 \\ -1 & -2-\lambda \end{bmatrix} = \lambda^2 + 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -5$$

Example 1 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -3-\lambda & -6 \\ -1 & -2-\lambda \end{bmatrix} = \lambda^2 + 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Example 1 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -3-\lambda & -6 \\ -1 & -2-\lambda \end{bmatrix} = \lambda^2 + 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

- Eigenvectors of A for $\lambda_2 = -5$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Example 1 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -3-\lambda & -6 \\ -1 & -2-\lambda \end{bmatrix} = \lambda^2 + 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

- Eigenvectors of A for $\lambda_2 = -5$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A + 5I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

- General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Example 1 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}, \ \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Example 1 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 1 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:
$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
- Use the initial condition:
$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 8/5 \end{bmatrix}$$

Example 1 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 8/5 \end{bmatrix}$$

- The solution to the initial value problem:

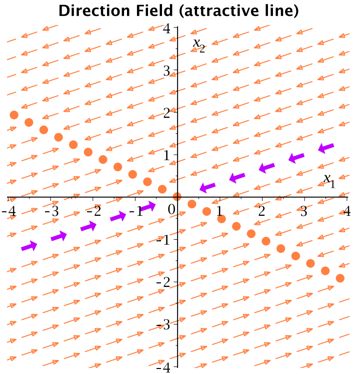
$$\vec{\mathbf{x}}(t) = \tfrac{7}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \tfrac{8}{5} e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

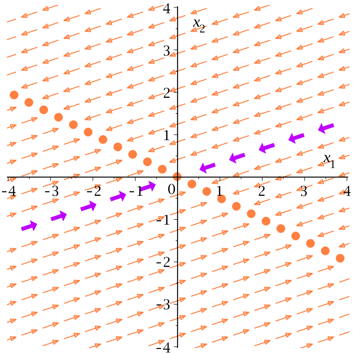


Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

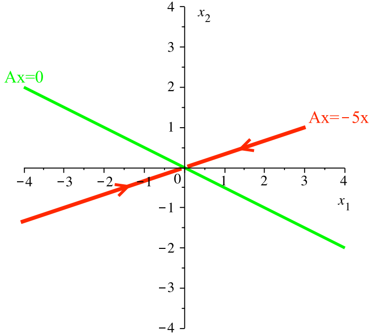
General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- $C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors of $\lambda_1 = 0$. These are equilibria.
- $C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ decays to the origin, along the eigenspace of $\lambda_2 = -5$.

Direction Field (attractive line)



Phase Portrait (attractive line of equilibria)

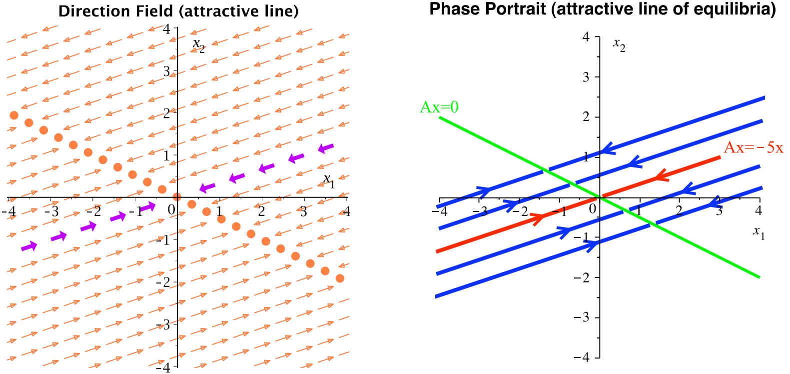


Example 1 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

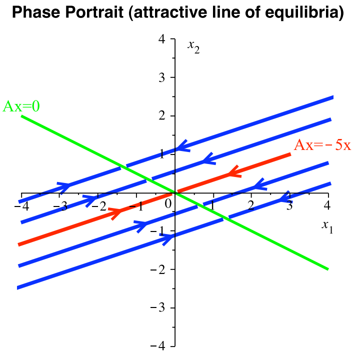
When C_1 and C_2 are both $\neq 0$: $\lim_{t\to\infty} \vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The solution trajectories are lines parallel to the eigenspace of $\lambda_2 = -5$.



Example 1 (d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

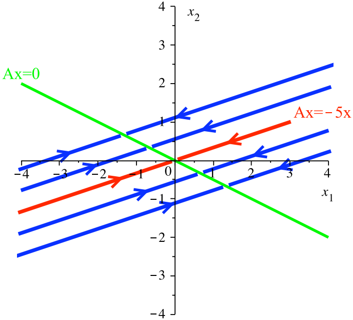
General solutions: $$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



Example 1 (d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

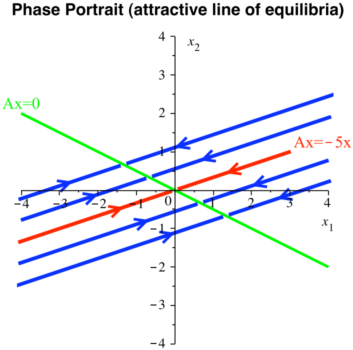
Phase Portrait (attractive line of equilibria)



Every equilibrium is stable,
but not asymptotically stable.

Example 1 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



Every equilibrium is
stable,
but not asymptotically stable.

We have an
attractive line of equilibria,
when $\lambda_1 = 0, \lambda_2 < 0$.

Example 2 (a repulsive line of equilibria).

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 2 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix} = \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 5$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix} = \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix} = \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- Eigenvectors of A for $\lambda_2 = 5$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 5I)\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Example 2 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix} = \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 5$$

- Eigenvectors of A for $\lambda_1 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- Eigenvectors of A for $\lambda_2 = 5$, by solving $(A - \lambda_2 I)\vec{\mathbf{x}} = 0$:

$$(A - 5I)\vec{\mathbf{x}} = 0 \quad \Leftrightarrow \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

- General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Example 2 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:
  $$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 7/5 \end{bmatrix}$$

Example 2 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 7/5 \end{bmatrix}$$

- The solution to the initial value problem:

$$\vec{\mathbf{x}}(t) = \tfrac{3}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \tfrac{7}{5} e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

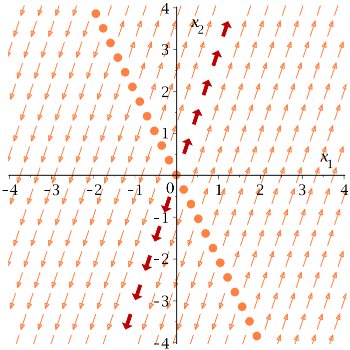
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Example 2 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

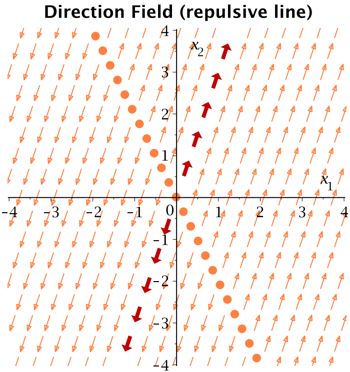
Direction Field (repulsive line)

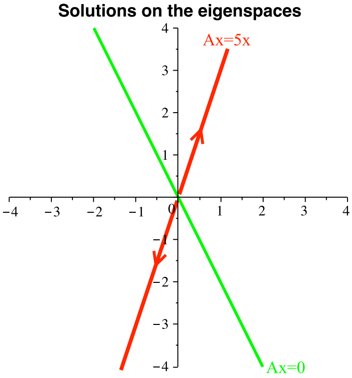


Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

- $C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors of $\lambda_1 = 0$. These are equilibria.
- $C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ grows, along the eigenspace of $\lambda_2 = 5$.



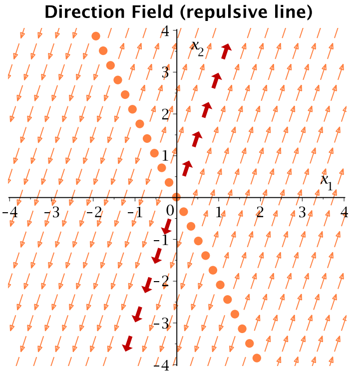


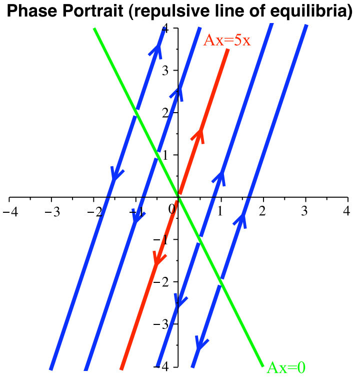
Example 2 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

When $C_1 \neq 0$ and $C_2 \neq 0$: All these solutions grow to infinity.

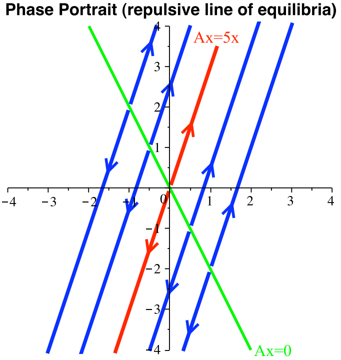
The trajectories are lines parallel to the eigenspace of $\lambda_2 = 5$.





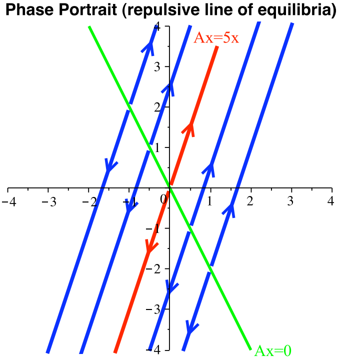
Example 2 (d) Are the equilibria stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



Example 2 (d) Are the equilibria stable, asymptotically stable, or unstable?

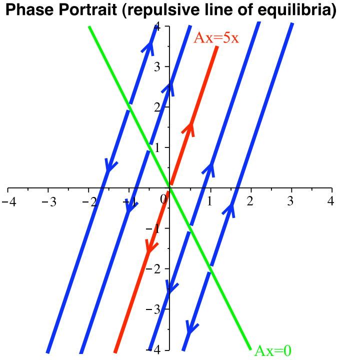
General solutions: $$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Every equilibrium is unstable.

Example 2 (d) Are the equilibria stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



Every equilibrium is unstable.

We have a *repulsive line of equilibria*, when $\lambda_1 = 0, \lambda_2 > 0$.