2D Homogeneous Linear Systems with Constant Coefficients — repeated eigenvalues

Xu-Yan Chen

Systems of Diff Eqs: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

where $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, A is a 2×2 real constant matrix

Things to explore:

- General solutions
- Initial value problems
- Geometric figures
  - Solutions graphs x_1 vs t & x_2 vs t
  - Direction fields in the (x_1, x_2) plane
  - Phase portraits in the (x_1, x_2) plane
- Stability/instability of equilibrium $(x_1, x_2) = (0, 0)$

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- Hard Cases: $A \neq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$, but $\lambda_1 = \lambda_2$.

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Find Solutions in the Easy Cases: $A = \lambda_1 I$

- All vector $\vec{\mathbf{x}} \in \mathbb{R}^2$ satisfy $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$.
  The eigenspace of λ_1 is the entire plane.
  We can pick $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linearly indep eigenvectors.

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  This gives partial solutions: $\vec{\mathbf{x}}(t) = Ce^{\lambda_1 t}\vec{\mathbf{u}}$.

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  Nonzero solutions of $(A - \lambda_1 I)^2\vec{\mathbf{x}} = 0$ are called *generalized eigenvectors*.

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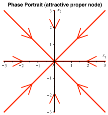
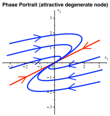
2D Systems $\vec{x}' = A\vec{x}$: phase portraits & stability

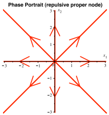
$\lambda_1 = \lambda_2 < 0, A = \lambda_1 I$:
Attractive proper node,
asymtotically stable



$\lambda_1 = \lambda_2 < 0, A \neq \lambda_1 I$:
Attractive degenerate node,
asymtotically stable



$\lambda_1 = \lambda_2 > 0, A = \lambda_1 I$:
Repulsive proper node,
unstable



$\lambda_1 = \lambda_2 > 0, A \neq \lambda_1 I$:
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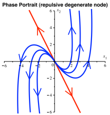
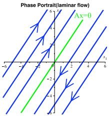


$\lambda_1 = \lambda_2 = 0, A = 0$:
Every point is a stable equilibrium, but not asymp stable

$\lambda_1 = \lambda_2 = 0, A \neq 0$:
Laminated flow,
unstable



Example 1. (Attractive Proper node)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

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- Eigenvalues of A are $\lambda_1 = \lambda_2 = -2$.
- $(A - \lambda_1 I)\vec{\mathbf{x}} = 0 \Leftrightarrow 0\vec{\mathbf{x}} = 0$: All $\vec{\mathbf{x}} \in \mathbb{R}^2$ are eigenvectors.
- General solutions are

$$\vec{\mathbf{x}}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad \vec{\mathbf{x}}(t) = e^{-2t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

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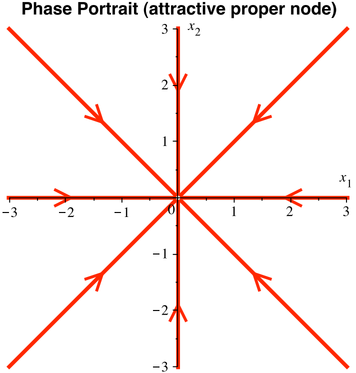
All solutions decay to 0 in the same exponential rate $\lambda_1 = -2$.
The trajectories are lines converging to the origin.



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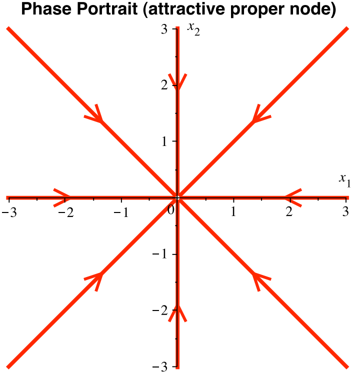

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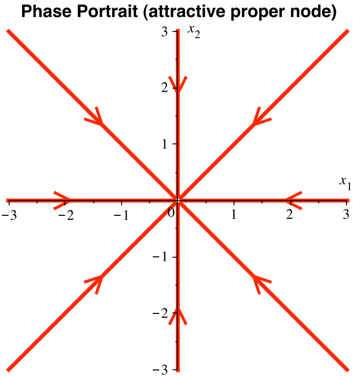

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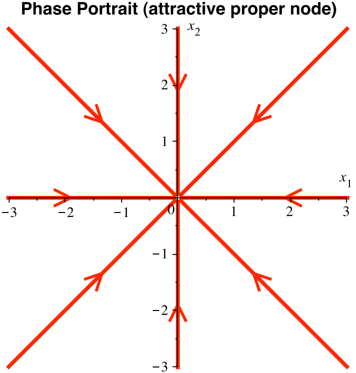

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The equilibrium $(0, 0)$ is asymptotically stable.

We have an *attractive proper node*, when $A = \lambda_1 I$ and $\lambda_1 < 0$.

Example 2. (Repulsive proper node)

$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

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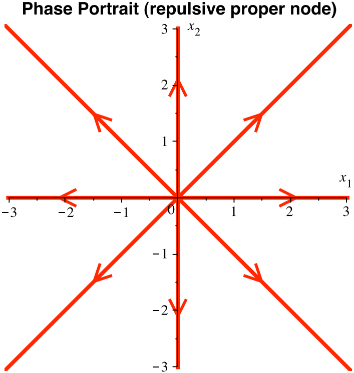
The trajectories are lines emanating from the origin.



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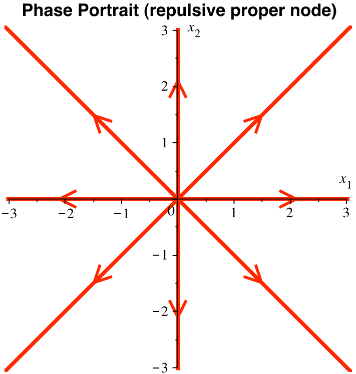

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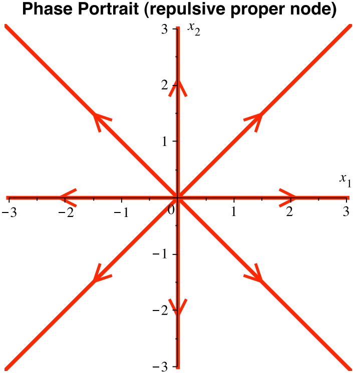

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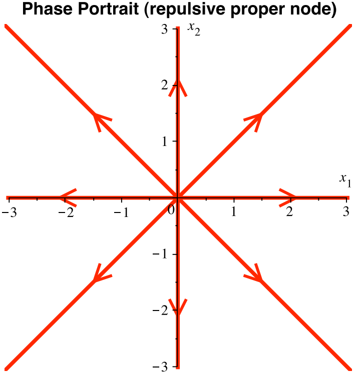

(d) Stability?

The equilibrium $(0, 0)$ is unstable.

We have a *repulsive proper node*, when $A = \lambda_1 I$ and $\lambda_1 > 0$.

Example 3. (attractive degenerate node)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 3 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

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- Eigenvectors of A for $\lambda_1 = \lambda_2 = -3$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A + 3I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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  - Can only pick one linear indep eigenvector $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
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Example 3 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

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$$\Leftrightarrow -2x_1 + 4x_2 = 1 \Leftrightarrow x_1 = -\tfrac{1}{2} + 2x_2 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2} + 2x_2 \\ x_2 \end{bmatrix}$$

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- General solutions are $\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}} + C_2 e^{\lambda_1 t}(\vec{\mathbf{v}} + t\vec{\mathbf{u}})$,

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- Find a generalized eigenvector, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = \vec{\mathbf{u}}$:

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- General solutions are $\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}} + C_2 e^{\lambda_1 t}(\vec{\mathbf{v}} + t\vec{\mathbf{u}})$,

$$\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

Example 3 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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$$\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

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- Use the initial condition:
$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

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- The solution to the initial value problem:

$$\vec{\mathbf{x}}(t) = 3e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 8e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = e^{-3t} \begin{bmatrix} 2 + 16t \\ 3 + 8t \end{bmatrix}$$

Example 3 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

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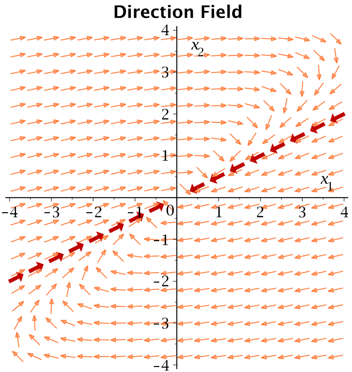

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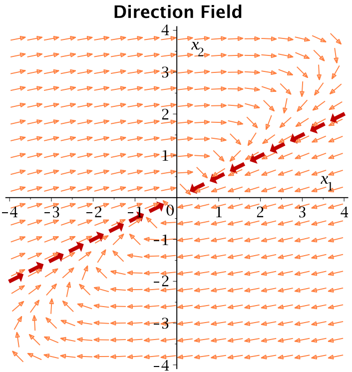
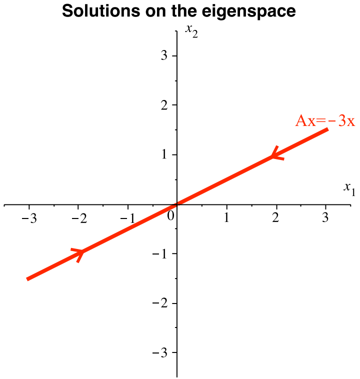
- When $C_2 = 0$, $\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ decays to the origin,

along the eigenspace of $\lambda_1 = \lambda_2 = -3$.





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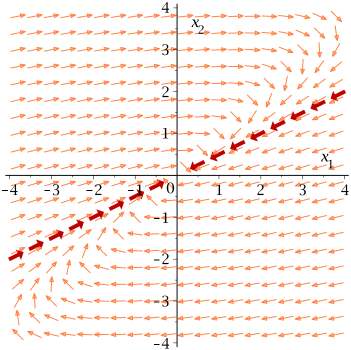
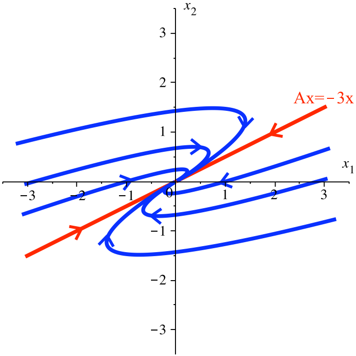
- When $C_2 \neq 0$, $\vec{\mathbf{x}}(t) \approx C_2 t e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $t \approx \infty$,

decaying to the origin along the eigenspace of $\lambda_1 = \lambda_2 = -3$.

Direction Field



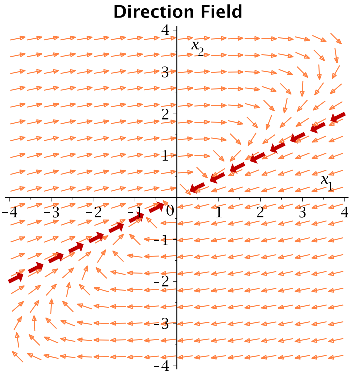
Phase Portrait (attractive degenerate node)

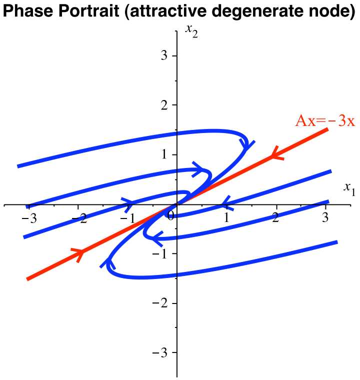


Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\quad \vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

- When $C_2 \neq 0$, $\vec{\mathbf{x}}(t) \approx C_2 t e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is very small, for $t \approx \infty$;





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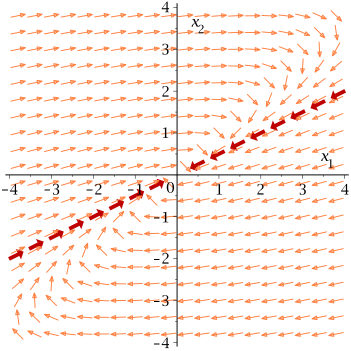
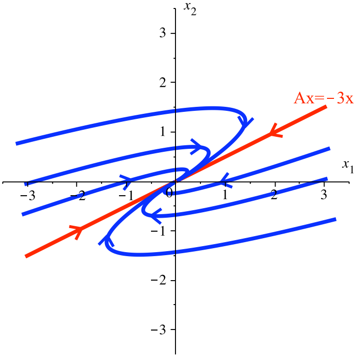


Phase Portrait (attractive degenerate node)



Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

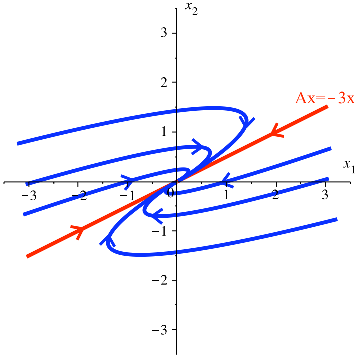
General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \left(\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

- When $C_2 \neq 0$, $\vec{\mathbf{x}}(t) \approx C_2 t e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is very small, for $t \approx \infty$;

  $\vec{\mathbf{x}}(t) \approx C_2 t e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is very large, for $t \approx -\infty$.

(d) Stability or instability?

Phase Portrait (attractive degenerate node)



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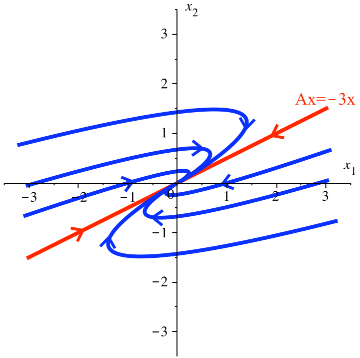
Phase Portrait (attractive degenerate node)



(d) Stability or instability?

The equilibrium $(0, 0)$ is asymptotically stable.

Example 3 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

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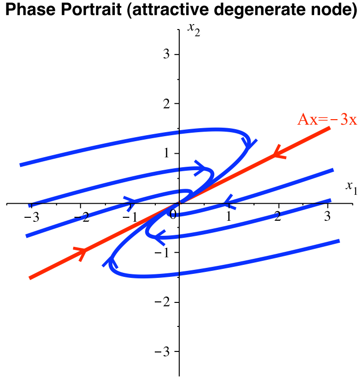

(d) Stability or instability?

The equilibrium $(0, 0)$ is asymptotically stable.

We have an *attractive degenerate node*, when $\lambda_1 = \lambda_2 < 0$, but $A \neq \lambda_1 I$.

Example 4. (repulsive degenerate node)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 4 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

Example 4 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 3$$

Example 4 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 3$$

- Eigenvectors of A for $\lambda_1 = \lambda_2 = 3$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A - 3I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Example 4 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 3$$

- Eigenvectors of A for $\lambda_1 = \lambda_2 = 3$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$(A - 3I)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

  - Can only pick one linear indep eigenvector $\vec{\mathbf{u}} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$.
  - Partial solutions: $\vec{\mathbf{x}}(t) = Ce^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$.
  - Need more to get complete solution formula.

Example 4 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = \lambda_2 = 3$
- An eigenvector for $\lambda_1 = \lambda_2 = 3$: $\vec{\mathbf{u}} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Example 4 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = \lambda_2 = 3$
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$$(A - 3I)\vec{\mathbf{x}} = \vec{\mathbf{u}} \Leftrightarrow \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\Leftrightarrow -2x_1 - x_2 = -\frac{1}{2} \Leftrightarrow x_1 = \frac{1}{4} - \frac{1}{2}x_2 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$$

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- General solutions are $\quad \vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}} + C_2 e^{\lambda_1 t}(\vec{\mathbf{v}} + t\vec{\mathbf{u}}),$

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- General solutions are $\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}} + C_2 e^{\lambda_1 t}(\vec{\mathbf{v}} + t\vec{\mathbf{u}})$,

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Example 4 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$$

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- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 14 \end{bmatrix}$$

Example 4 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

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$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 14 \end{bmatrix}$$

- The solution to the initial value problem:

$$\vec{\mathbf{x}}(t) = 3e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + 14e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right) = e^{3t} \begin{bmatrix} 2 - 7t \\ 3 + 14t \end{bmatrix}$$

Example 4 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$

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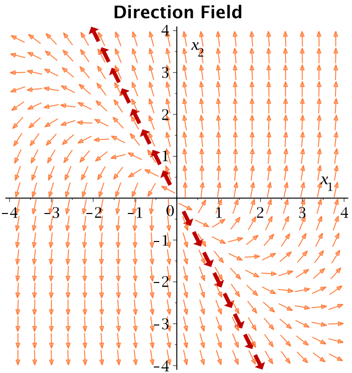
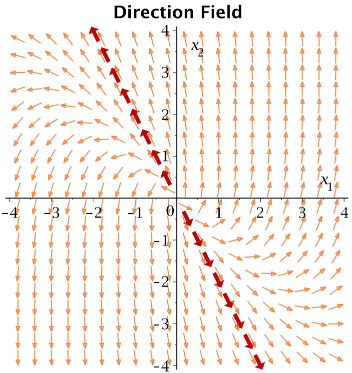


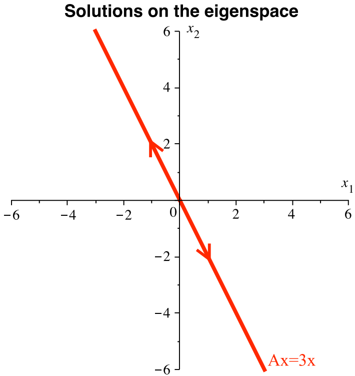
Example 4 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$

- When $C_2 = 0$, $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ leaves the origin,

along the eigenspace of $\lambda_1 = \lambda_2 = 3$.





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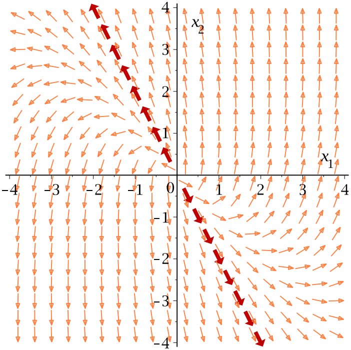
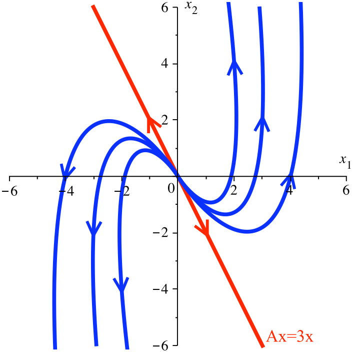
- When $C_2 \neq 0$, $\vec{\mathbf{x}}(t) \approx C_2 t e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is very large, for $t \approx \infty$;

  $\vec{\mathbf{x}}(t) \approx C_2 t e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is very small, for $t \approx -\infty$.

Direction Field



Phase Portrait (repulsive degenerate node)



Example 4 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$

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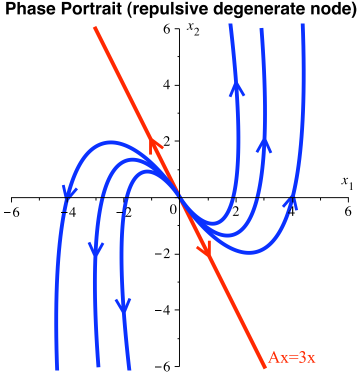

(d) Stability or instability?

Example 4 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$

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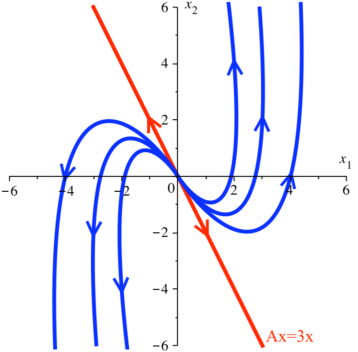
(d) Stability or instability?

The equilibrium $(0, 0)$ is unstable.

Phase Portrait (repulsive degenerate node)



Example 4 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix} \vec{\mathbf{x}}$

General solutions: $\vec{\mathbf{x}}(t) = C_1 e^{3t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$

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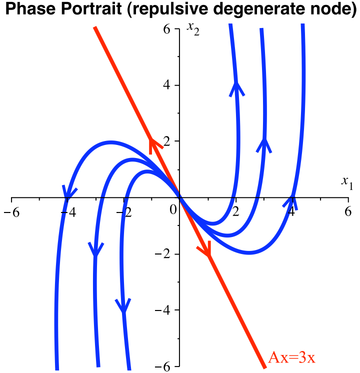
(d) Stability or instability?

The equilibrium $(0,0)$ is unstable.

We have a *repulsive degenerate node*, when $\lambda_1 = \lambda_2 > 0$, but $A \neq \lambda_1 I$.



Example 5. (laminated flow)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 5 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

Example 5 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -6-\lambda & 4 \\ -9 & 6-\lambda \end{bmatrix} = \lambda^2 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 0$$

Example 5 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

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$$\det \begin{bmatrix} -6-\lambda & 4 \\ -9 & 6-\lambda \end{bmatrix} = \lambda^2 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 0$$

- Eigenvectors of A for $\lambda_1 = \lambda_2 = 0$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$A\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Example 5 (a) $\vec{\mathbf{x}}' = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

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  - Can only pick one linear indep eigenvector $\vec{\mathbf{u}} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$.
  - Equilibrium solutions: $\vec{\mathbf{x}}(t) = C \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$.
  - Need more to get complete solution formula.

Example 5 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = \lambda_2 = 0$
- An eigenvector for $\lambda_1 = \lambda_2 = 0$: $\vec{\mathbf{u}} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

Example 5 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = \lambda_2 = 0$
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- Find a generalized eigenvector, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = \vec{\mathbf{u}}$:

$$A\vec{\mathbf{x}} = \vec{\mathbf{u}} \Leftrightarrow \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

$$\Leftrightarrow -6x_1 + 4x_2 = \tfrac{2}{3} \Leftrightarrow x_1 = -\tfrac{1}{9} + \tfrac{2}{3}x_2 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} + \frac{2}{3}x_2 \\ x_2 \end{bmatrix}$$

Example 5 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A: $\lambda_1 = \lambda_2 = 0$
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$\Rightarrow$ A generalized eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}$.

Example 5 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

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$$\Rightarrow \text{A generalized eigenvector } \vec{\mathbf{v}} = \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}.$$

- General solutions are $\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}} + C_2 e^{\lambda_1 t}(\vec{\mathbf{v}} + t\vec{\mathbf{u}})$,

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$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right)$$

Example 5 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Example 5 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right)$$

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- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{2}{3} & -\frac{1}{9} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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- The solution to the initial value problem:

$$\vec{\mathbf{x}}(t) = 3 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{(an equilibrium)}$$

Example 5 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right)$$

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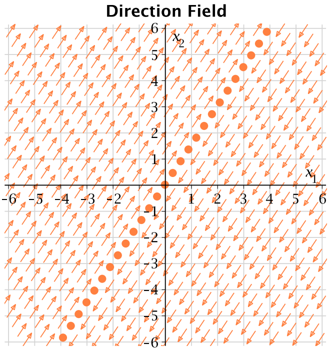
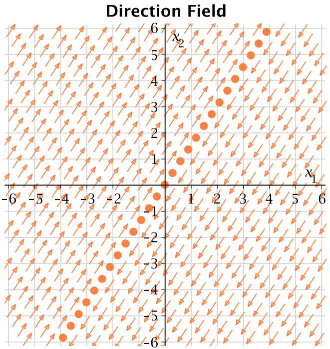


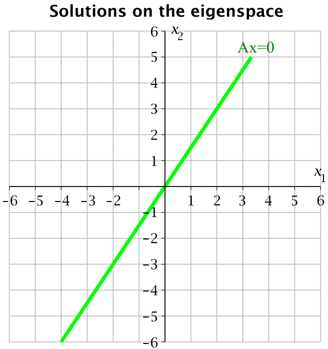
Example 5 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right)$$

- When $C_2 = 0$: $\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ are equilibria, lining along the eigenspace.





Example 5 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

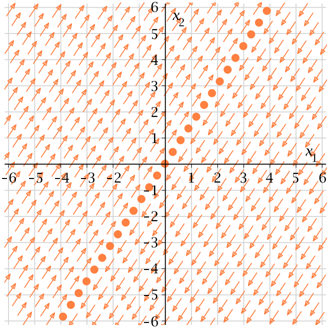
General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) = \left(C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} \right) + t\, C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

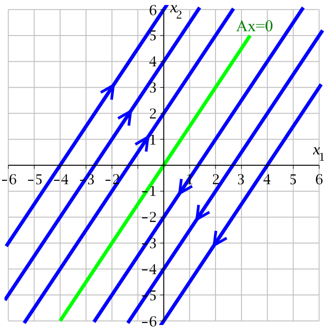
- When $C_2 \neq 0$: $\vec{\mathbf{x}}(t)$ are linear functions, with $\vec{\mathbf{x}}(0) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}$;

  velocity $\dfrac{d\vec{\mathbf{x}}}{dt} = C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ is parallel to the eigenspace.

Direction Field



Phase Portrait (laminar flow)



Example 5 (c) Phase portrait of $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

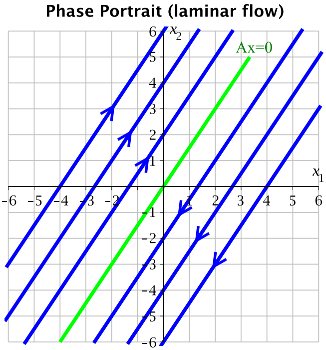
General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) = \left(C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} \right) + t\, C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- When $C_2 \neq 0$: $\vec{\mathbf{x}}(t)$ are linear functions, with $\vec{\mathbf{x}}(0) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}$;

  velocity $\frac{d\vec{\mathbf{x}}}{dt} = C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ is parallel to the eigenspace.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?



Example 5 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) = \left(C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} \right) + t\, C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- When $C_2 \neq 0$: $\vec{\mathbf{x}}(t)$ are linear functions, with $\vec{\mathbf{x}}(0) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}$;

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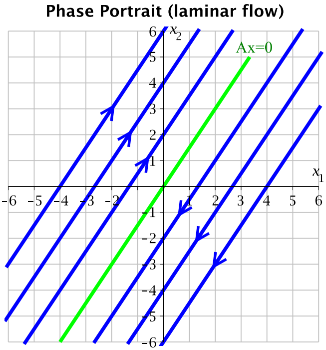
The equilibrium $(0, 0)$ is unstable.



Example 5 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix} \vec{\mathbf{x}}$

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$$\vec{\mathbf{x}}(t) = C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) = \left(C_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix} \right) + t\, C_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

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(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

The equilibrium $(0, 0)$ is unstable.

We have a *laminar flow*, when $\lambda_1 = \lambda_2 = 0$, but $A \neq 0$.