2D Homogeneous Linear Systems with Constant Coefficients — complex eigenvalues

Xu-Yan Chen

Systems of Diff Eqs: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

where $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, A is a 2×2 real constant matrix

Things to explore:

- General solutions
- Initial value problems
- Geometric figures
  - Solutions graphs x_1 vs t & x_2 vs t
  - Direction fields in the (x_1, x_2) plane
  - Phase portraits in the (x_1, x_2) plane
- Stability/instability of equilibrium $(x_1, x_2) = (0, 0)$

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  - Recall Euler's formula:

    $$e^{\lambda_1 t} = e^{(\alpha+\beta i)t} = e^{\alpha t}e^{\beta i t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)].$$

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- General solutions are
  $$\vec{\mathbf{x}}(t) = C_1\,\text{Re}\left(e^{\lambda_1 t}\vec{\mathbf{u}}_1\right) + C_2\,\text{Im}\left(e^{\lambda_1 t}\vec{\mathbf{u}}_1\right).$$

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General Solution Formula:
Assume that A has a complex eigenvalue $\lambda_1 = \alpha + \beta i$ and a corresponding eigenvector $\vec{\mathbf{u}}_1 = \vec{\mathbf{a}} + i\,\vec{\mathbf{b}}$.
(It follows that $\lambda_2 = \alpha - \beta i$ is the other eigenvalue and $\vec{\mathbf{u}}_2 = \vec{\mathbf{a}} - i\,\vec{\mathbf{b}}$ is its eigenvector.)

- *Complex-valued formula*:

  $$\vec{\mathbf{x}}(t) = C_1 e^{\lambda_1 t}\vec{\mathbf{u}}_1 + C_2 e^{\lambda_2 t}\vec{\mathbf{u}}_2.$$

- *Real-valued formula*:

  $$\vec{\mathbf{x}}(t) = C_1 \operatorname{Re}\left(e^{\lambda_1 t}\vec{\mathbf{u}}_1\right) + C_2 \operatorname{Im}\left(e^{\lambda_1 t}\vec{\mathbf{u}}_1\right).$$

- *Real-valued formula (expanded)*:

  $$\vec{\mathbf{x}}(t) = C_1 e^{\alpha t}\Big(\cos(\beta t)\vec{\mathbf{a}} - \sin(\beta t)\vec{\mathbf{b}}\Big) + C_2 e^{\alpha t}\Big(\sin(\beta t)\vec{\mathbf{a}} + \cos(\beta t)\vec{\mathbf{b}}\Big).$$

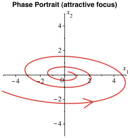
Note: $\alpha = \operatorname{Re}\lambda_1$ gives the growth/decay rate,
$\beta = \operatorname{Im}\lambda_1$ is the frequency of the oscillation.

2D Systems: $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$

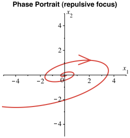
Phase portraits & stability of the equilibrium $(0,0)$:

Assume that A has complex eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$.

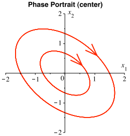
- $\alpha = \operatorname{Re}\lambda_1 < 0$
  $\Rightarrow$ Attractive focus,
  asymtotically stable



- $\alpha = \operatorname{Re}\lambda_1 > 0$
  $\Rightarrow$ Repulsive focus,
  unstable



- $\alpha = \operatorname{Re}\lambda_1 = 0$
  $\Rightarrow$ Center,
  stable,
  but not asymptotically stable



Example 4. (Complex eigenvalues)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 1 & \frac{5}{2} \\ -2 & -1 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & \frac{5}{2} \\ -2 & -1 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & \frac{5}{2} \\ -2 & -1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

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- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

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$$\vec{\mathbf{x}}(t) = C_1 \left(\cos(2t) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} - \sin(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + C_2 \left(\sin(2t) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + \cos(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

Example 4 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & \frac{5}{2} \\ -2 & -1 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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- The solution to the initial value problem:

$$\begin{aligned}\vec{\mathbf{x}}(t) = \quad & 2 \left(\cos(2t) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} - \sin(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\ & -2 \left(\sin(2t) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + \cos(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right),\end{aligned}$$

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} \cos(2t) + 3\sin(2t) \\ 2\cos(2t) - 2\sin(2t) \end{bmatrix}$$

Example 4 (c) Phase portrait of $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & \frac{5}{2} \\ -2 & -1 \end{bmatrix} \vec{\mathbf{x}}$

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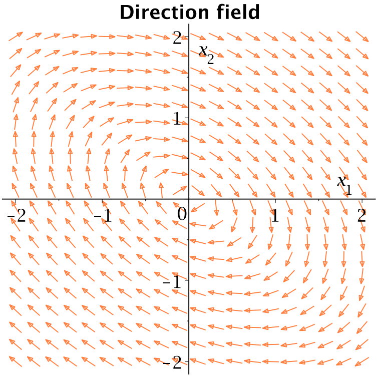

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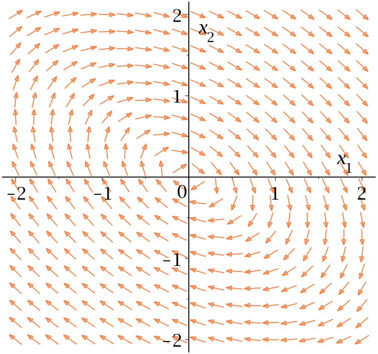
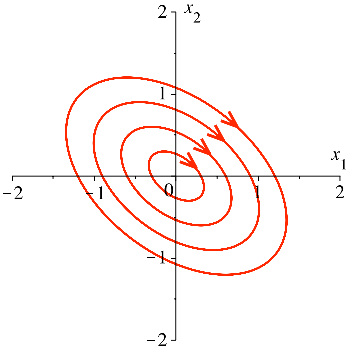
Periodic oscillations: frequency $= 2$, period $= 2\pi/2 = \pi$.

Direction field



Phase Portrait (center)



Example 4 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

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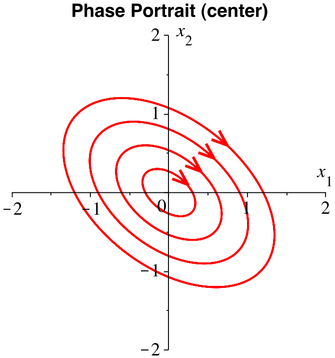

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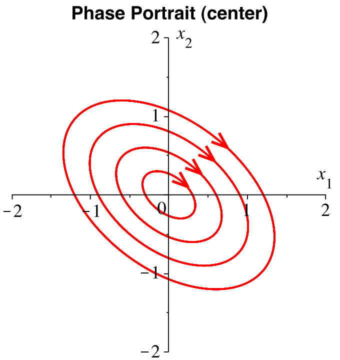
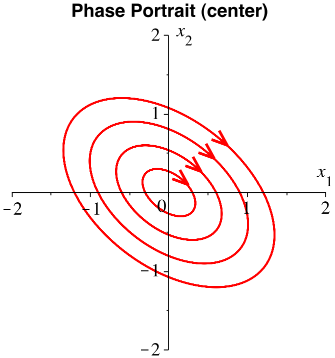


Solutions starting near $(0,0)$ stay close to $(0,0)$, but $\lim_{t\to\infty} \vec{\mathbf{x}}(t) \neq (0,0)$.

Example 4 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $\vec{\mathbf{x}}(t) = C_1\left(\cos(2t)\begin{bmatrix}-1/2\\1\end{bmatrix} - \sin(2t)\begin{bmatrix}-1\\0\end{bmatrix}\right) + C_2\left(\sin(2t)\begin{bmatrix}-1/2\\1\end{bmatrix} + \cos(2t)\begin{bmatrix}-1\\0\end{bmatrix}\right)$



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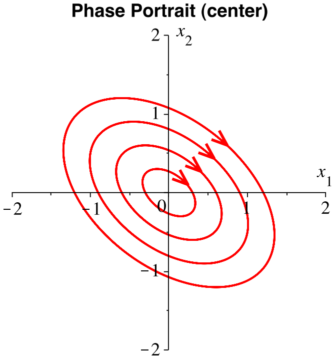

Solutions starting near $(0,0)$ stay close to $(0,0)$, but $\lim_{t\to\infty} \vec{\mathbf{x}}(t) \neq (0,0)$.

The equilibrium $(0,0)$ is stable, but not asymptotically stable.

We have a *center*, when eigenvalues $\lambda = \pm\beta i$.

Example 5. (Complex eigenvalues)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 5 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

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- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -5-\lambda & -39 \\ 6 & 1-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 229 = 0$$

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- Eigenvectors of A for $\lambda_1 = -2 + 15i$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

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- Eigenvectors of A for $\lambda_1 = -2 + 15i$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$\Big(A - (-2 + 15i)I\Big)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -3-15i & -39 \\ 6 & 3-15i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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- General solutions:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & C_1 e^{-2t} \left(\cos(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right) \\ & + C_2 e^{-2t} \left(\sin(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right) \end{aligned}$$

Example 5 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 5 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{4}{5} \end{bmatrix}$$

Example 5 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

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- The solution to the initial value problem:

$$\begin{aligned} \vec{\mathbf{x}}(t) = &\quad 2e^{-2t}\left(\cos(15t)\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin(15t)\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right) \\ &+ \tfrac{4}{5} e^{-2t}\left(\sin(15t)\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos(15t)\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right), \end{aligned}$$

$$\vec{\mathbf{x}}(t) = e^{-2t}\begin{bmatrix} \cos(15t) - \frac{27}{5}\sin(15t) \\ 2\cos(15t) + \frac{4}{5}\sin(15t) \end{bmatrix}$$

Example 5 (c) Phase portrait of $\vec{\mathbf{x}}' = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

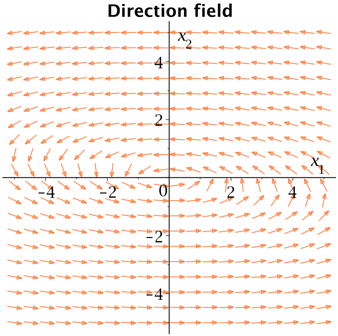
General solutions:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & C_1 e^{-2t} \left(\cos(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right) \\ & + C_2 e^{-2t} \left(\sin(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right) \end{aligned}$$

Example 5 (c) Phase portrait of $\vec{\mathbf{x}}' = \begin{bmatrix} -5 & -39 \\ 6 & 1 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^{-2t}\left(\cos(15t)\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin(15t)\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right) + C_2 e^{-2t}\left(\sin(15t)\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos(15t)\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}\right)$$



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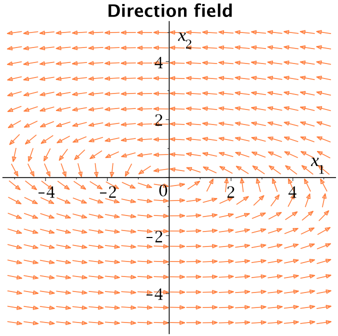
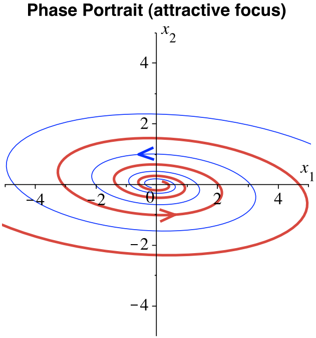
General solutions:

$$\vec{\mathbf{x}}(t) = \quad C_1 e^{-2t} \left(\cos(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} - \sin(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right)$$

$$+ C_2 e^{-2t} \left(\sin(15t) \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \cos(15t) \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} \right)$$

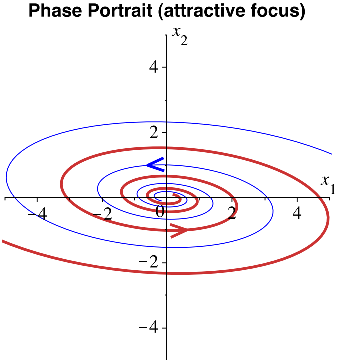
Decaying oscillations: $\begin{cases} \text{decay rate} = -2 = \operatorname{Re} \lambda, \\ \text{frequency} = 15 = \operatorname{Im} \lambda \end{cases}$





Example 5 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions:
$$\vec{\mathbf{x}}(t) = C_1 e^{-2t}\left(\cos(15t)\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix} - \sin(15t)\begin{bmatrix}\frac{5}{2}\\ 0\end{bmatrix}\right) + C_2 e^{-2t}\left(\sin(15t)\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix} + \cos(15t)\begin{bmatrix}\frac{5}{2}\\ 0\end{bmatrix}\right)$$



The equilibrium $(0,0)$ is asymptotically stable.

We have an *attractive focus*, when complex eigenvalues $\lambda = \alpha \pm \beta i$ have $\operatorname{Re}\lambda = \alpha < 0$.

Example 6. (Complex eigenvalues)

Consider $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix}$.

(a) Find general solutions of $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Sketch the phase portrait.

(d) Is the equilibrium $(0, 0)$ stable, asymptotically stable, or unstable?

Example 6 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

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- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -1-\lambda & 10 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

Example 6 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

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$$\Rightarrow \lambda_1 = 1 + 4i, \lambda_2 = 1 - 4i$$

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- Eigenvectors of A for $\lambda_1 = 1 + 4i$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

Example 6 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -1-\lambda & 10 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

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- Eigenvectors of A for $\lambda_1 = 1 + 4i$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$\Big(A - (1+4i)I\Big)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -2-4i & 10 \\ -2 & 2-4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 6 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

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$$\Big(A - (1+4i)I\Big)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -2-4i & 10 \\ -2 & 2-4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow -2x_1 + (2-4i)x_2 = 0$$

Example 6 (a) $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

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$$\Leftrightarrow -2x_1 + (2-4i)x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Example 6 (a) $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

- Eigenvalues of A, by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -1-\lambda & 10 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

$$\Rightarrow \lambda_1 = 1 + 4i, \lambda_2 = 1 - 4i$$

- Eigenvectors of A for $\lambda_1 = 1 + 4i$, by solving $(A - \lambda_1 I)\vec{\mathbf{x}} = 0$:

$$\Big(A - (1+4i)I\Big)\vec{\mathbf{x}} = 0 \Leftrightarrow \begin{bmatrix} -2-4i & 10 \\ -2 & 2-4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow -2x_1 + (2-4i)x_2 = 0 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{An eigenvector } \vec{\mathbf{u}}_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right)$$

Example 6 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 6 (b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \\ & + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Example 6 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \\ & + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \end{aligned}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

Example 6 (b) Solve $\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}, \ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- General solutions:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \\ & + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \end{aligned}$$

- Use the initial condition:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

- The solution to the initial value problem:

$$\begin{aligned} \vec{\mathbf{x}}(t) = \quad & 2e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) \\ & + \tfrac{1}{2} e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right), \end{aligned}$$

$$\vec{\mathbf{x}}(t) = e^t \begin{bmatrix} \cos(4t) + \frac{9}{2} \sin(4t) \\ 2\cos(4t) + \frac{1}{2} \sin(4t) \end{bmatrix}$$

Example 6 (c) Phase portrait of $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

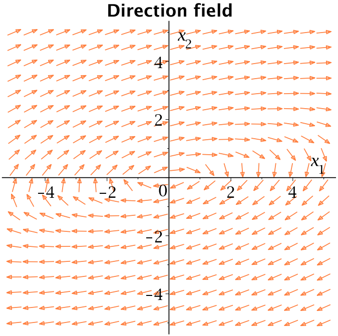
General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right)$$

Example 6 (c) Phase portrait of $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right)$$

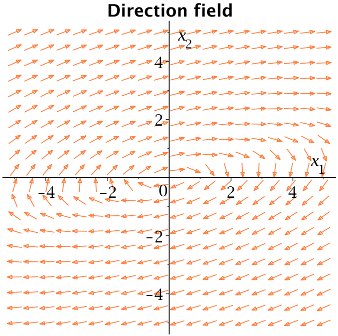


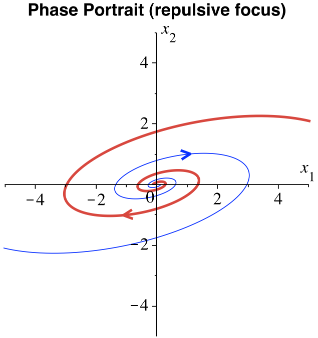
Example 6 (c) Phase portrait of $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{\mathbf{x}}$

General solutions:

$$\vec{\mathbf{x}}(t) = C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right)$$

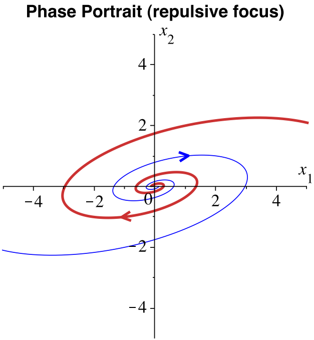
Growing oscillations: $\begin{cases} \text{growth rate} = 1 = \operatorname{Re}\lambda, \\ \text{frequency} = 4 = \operatorname{Im}\lambda \end{cases}$





Example 6 (d) Is the equilibrium $(0,0)$ stable, asymptotically stable, or unstable?

General solutions: $$\vec{\mathbf{x}}(t) = C_1 e^t \left(\cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) + C_2 e^t \left(\sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right)$$



The equilibrium $(0,0)$ is unstable.

We have a *repulsive focus*, when complex eigenvalues $\lambda = \alpha \pm \beta i$ have $\operatorname{Re} \lambda = \alpha > 0$.