Separable Diff. Equations

$$\frac{dy}{dx} = f(x)\,g(y).$$

Examples

- $\dfrac{dy}{dx} = e^{2x}(1+y)$
- $\dfrac{dy}{dx} = y - y^2$
- $\dfrac{dy}{dx} = e^{-3y}\sin(4x)$
- $\dfrac{dy}{dx} = x^3y^2$
- $\dfrac{dy}{dx} = e^{-x+2y} \quad (= e^{-x}\cdot e^{2y})$
- $\dfrac{dy}{dx} = xy^4 + y^4 \quad (= (x+1)y^4)$

Non-Examples

- $\dfrac{dy}{dx} = e^{2x} + y$
- $\dfrac{dy}{dx} = xy - y^2$

How to Solve Separable Diff Equations?

$$\frac{dy}{dx} = f(x)\, g(y)$$

Solution Method: (Separate variables)

$$\Rightarrow \frac{1}{g(y)}\, dy = f(x)\, dx$$

$$\Rightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$\Rightarrow$ An equation relating x & y

If y is solvable from this eq, go ahead solve it.

Example 1 Solve $\frac{dy}{dx} = e^{2y}\cos x$.

Solution:

$$e^{-2y}\,dy = \cos x\,dx.$$

$$\int e^{-2y}\,dy = \int \cos x\,dx$$

$$-\frac{1}{2}e^{-2y} = \sin x + C \qquad (*)$$

$$e^{-2y} = -2\sin x - 2C$$

$$-2y = \ln(-2\sin x - 2C)$$

$$\boxed{y = -\frac{1}{2}\ln(-2\sin x - 2C)}$$

Example 2 Solve $\frac{dy}{dx} = e^{2y}\cos x$, $y(0) = -\ln 3$.

Solution: Diff Eq. solved in Example 1:

$$y = -\frac{1}{2}\ln(-2\sin x - 2C).$$

Init. Cond. $\Rightarrow -\ln 3 = -\frac{1}{2}\ln(-2C)$

$\Rightarrow \ln(-2C) = 2\ln 3 = \ln 9$

$\Rightarrow -2C = 9, \quad C = -\frac{9}{2}$ Can also be obtained from (*)

$$\Rightarrow \boxed{y = -\frac{1}{2}\ln(-2\sin x + 9)}$$

Example 3. Solve $\frac{dy}{dt} = y^2$.

Solution

$$\frac{1}{y^2}\,dy = dt$$

$$\int \frac{1}{y^2}\,dy = \int dt$$

$$-\frac{1}{y} = t + C \qquad (*)$$

$$\boxed{y = -\frac{1}{t+C}} \qquad (**)$$

Example 4. Solve $\frac{dy}{dt} = y^2$, $y(0) = -\frac{1}{3}$.

Solution Diff Eq. $\xRightarrow{\text{Example 3}}$ $y = -\frac{1}{t+C}$

Init. Cond. $\Rightarrow$ $-\frac{1}{3} = -\frac{1}{C}$ $\Rightarrow$ $C = 3$

$$\boxed{y = -\frac{1}{t+3}}$$

Solution 2 In eq. (*), use the init. cond. $y(0) = -\frac{1}{3}$:

$$-\frac{1}{-1/3} = 0 + C \Rightarrow C = 3$$

Plug in (**) $\Rightarrow$ $\boxed{y = -\frac{1}{t+3}}$