Scalar Separable Equations: $\dfrac{dy}{dx} = f(x)g(y)$

Solution Method: Rewrite the given equation (symbolically):

$$\frac{dy}{g(y)} = f(x)dx.$$

Now integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

This gives a realtion between x and y. If you can solve this equation for y in terms of x, go ahead.

Example: Solve the initial value problem $\dfrac{dy}{dx} = e^{2y}\cos x, y(0) = -\dfrac{1}{3}$.

Solution: Rewrite the diff. eq. into $e^{-2y}dy = \cos x dx$ and then integrate:

$$\int e^{-2y}dy = \int \cos x dx.$$

This gives

$$-\frac{1}{2}e^{-2y} = \sin x + C,$$

where C is an arbitray constant. Solve the last equation for y in terms of x:

$$y = -\frac{1}{2}\ln(-2\sin x - 2C).$$

This is the general solutions to the given ODE. Now examine the initial condition:

$$y(0) = -\frac{1}{3} \qquad \Longrightarrow -\frac{1}{3} = -\frac{1}{2}\ln(-2C) \qquad \Longrightarrow C = -\frac{1}{2}e^{2/3}.$$

Thus, the answer to the problem is

$$y = -\frac{1}{2}\ln(-2\sin x + e^{2/3}).$$

Exercises

Solve the following ODEs and initial value problems of ODEs.

[1] $y'(t) + (2t - \sin 2t)y(t) = 0$

[2] $tx'(t) + 4x(t) = 0 \ (t > 0), \ \ x(3) = 2$

[3] $y'(x) = x^2 e^x y(x)^2$

[4] $\frac{dy}{dx} = (1 + 6x)y(1 - y), y(0) = 1/3$

Answers

[1] $y = Ce^{-t^2 - 0.5\cos 2t}$

[2] $x(t) = 162t^{-4}$

[3] $y = -1/(C + x^2 e^x - 2xe^x + 2e^x)$

[4] $y = \dfrac{e^{x+3x^2}}{2 + e^{x+3x^2}}$