First order scalar linear equations: $\dfrac{dy}{dx} + a(x)y = b(x)$

Solution Method: (The Method of Integrating Factor)

Step 1. Find an antiderivative $A(x)$ of $a(x)$; namely, $A(x)$ satisfies $A'(x) = a(x)$. Such a function $A(x)$ can be found by integrating $a(x)$.

Step 2. Multiply both sides of the ODE by $e^{A(x)}$:

$$e^{A(x)}\left(\frac{dy}{dx} + a(x)y\right) = b(x)e^{A(x)}.$$

Notice that the left hand side equals $\frac{d}{dx}\left(e^{A(x)}y\right)$. We obtain

$$\frac{d}{dx}\left(e^{A(x)}y\right) = b(x)e^{A(x)}.$$

Now integrating the two sides yields:

$$e^{A(x)}y = \int b(x)e^{A(x)}dx + C.$$

Thus,

$$y = e^{-A(x)}\int b(x)e^{A(x)}dx + Ce^{-A(x)}.$$

Example: Solve $y' + 2xy = x^3$.

Solution: Here, $a(x) = 2x$. The function $A(x) = x^2$ is an antiderivative of $a(x)$. Multiply both sides of the ODE by $e^{A(x)} = e^{x^2}$:

$$e^{x^2}(y' + 2xy) = x^3e^{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(e^{x^2}y\right) = x^3e^{x^2}.$$

An integration gives

$$e^{x^2}y = \int x^3e^{x^2}dx + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

Thus,

$$y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2},$$

where C is an arbitrary constant.

Exercises

Solve the following ODEs and initial value problems of ODEs.

[1] $y'(t) + (2t - \sin 2t)y(t) = 0$

[2] $tx'(t) + 4x(t) = 0 \ (t > 0), \ \ x(3) = 2$

[3] $y' + 2y = e^x$

[4] $y'(t) + (2t - \sin 2t)y(t) = 3e^{-t^2} \sin 2t$

[5] $y' + (6x^2 - 2x)y = 150x^5 e^{3x^3 + x^2}$

[6] $y' + \dfrac{2}{x} y = x^5, y(1) = 1$

Answers

[1] $y = Ce^{-t^2 - 0.5\cos 2t}$

[2] $x(t) = 162t^{-4}$

[3] $y = \dfrac{1}{3}e^x + Ce^{-2x}$

[4] $y = -3e^{-t^2} + Ce^{-t^2 - 0.5\cos 2t}$

[5] $y = 2(5x^3 - 1)e^{3x^3 + x^2} + Ce^{-2x^3 + x^2}$

[6] $y = \dfrac{1}{8}x^6 + \dfrac{7}{8}x^{-2}$