Differences between the Solution Structures of Linear and Nonlinear Equations

[1] Given the fact that $y_p = 0$ is a particular solution of a homogeneous linear equation $y' = f(x)y$, can you find all solutions of this differential equation?

[2] Given the fact that $y_p = -5e^{3\sin 2x}$ is a particular solution of a homogeneous linear equation $y' = f(x)y$, can you find all solutions of this differential equation?

[3] Given a homogeneous linear equation $y' = f(x)y$, is it possible that both $y_1 = e^{2x}$ and $y_2 = e^x$ are particular solutions of the same equation $y' = f(x)y$?

[4] Given the fact that $y_p = x^2$ is a particular solution of a nonhomogeneous linear equation $y' = f(x)y + p(x)$, can you find all solutions of this differential equation?

[5] Given the facts that

$$\begin{cases} y_p = x^2 \text{ is a particular solution of a nonhomogeneous linear equation } y' = f(x)y + p(x), \\ \text{and } y_0 = -5e^{3\sin 2x} \text{ is a particular solution of the homogeneous linear equation } y' = f(x)y, \end{cases}$$

can you find all solutions of the nonhomogeneous linear equation $y' = f(x)y + p(x)$?

[6] Given the fact that $y_1 = e^x$ and $y_2 = -2e^{-x}$ both satisfy the same nonhomogeneous linear equation $y' = f(x)y + p(x)$, can you find all solutions of the nonhomogeneous linear equation $y' = f(x)y + p(x)$?

[7] Given the fact that $y_1 = e^x$ is a solution of a nonlinear differential equation $y' = f(x)g(y)$, can you find all solutions of this equation?

[8] Given the fact that $y_1 = e^x$ and $y_2 = -2e^{-x}$ are solutions of a nonlinear differential equation $y' = f(x)g(y)$, can you find all solutions of this equation?

[9] Given the fact that all four functions $y_1 = e^x$, $y_2 = -2e^{-x}$, $y_3 = 3e^x$, and $y_4 = -3e^{-x}$ are particular solutions of a nonlinear differential equation $y' = f(x, y)$, can you find all solutions of this equation?

(See next page for answers)

Answers:

[1] Not enough information.

[2] $y = Ce^{3\sin 2x}$ where C is a free parameter.

[3] Impossible.

[4] Not enough information.

[5] $y = x^2 + Ce^{3\sin 2x}$ where C is a free parameter.

[6] $y = e^x + C(e^x + 2e^{-x})$ where C is a free parameter.
Hint: If y_1 and y_2 satisfy the same nonhomegeneous equation $y' = f(x)y + p(x)$, then $y_1 - y_2$ will be a solution of the corresponding homogeneous equation $y' = f(x)y$.

[7] Not enough information.

[8] Not enough information.

[9] Not enough information.

Comment: Almost no calculation is necessary in solving the above problems. If you were doing some involved computations, you were on wrong track. You should be able to solve each of them in a few seconds. (The only exception might be [6], in which you need to observe the property in the above hint that may take you some time.)