A Very Brief Intro to Diff Eq's

Algebraic Equations unknowns are numbers

Example Solve (*) $x^2+5x+6=0$.

i.e. Find all numbers x satisfying (*).

Differential Equations unknowns are functions

Example Solve (**) $\frac{dy}{dt} = \frac{1}{3}y$

i.e. Find all functions $y(t)$ satisfying (**)

Example Solve (***) $\frac{dy}{dt} + 2ty = t^3$

i.e. Find all functions $y(t)$ satisfying (***)

Question Solve diff eq $\frac{dy}{dt} = \frac{1}{3}y$

i.e. Find all functions $y(t)$ satisfying this eq.

Solve $\begin{cases} \frac{dy}{dt} = \frac{1}{3} y & \text{(Diff Eq)} \\ y(0) = 7 & \text{(Initial Condition)} \end{cases}$

↗ an initial value problem of diff eq

Solution:

- Diff Eq ⇒ General Solutions $y(t) = ce^{\frac{1}{3}t}$
- Initial Cond. ⇒ $7 = ce^{0}$ ⇒ $c = 7$

$$\boxed{y(t) = 7e^{\frac{1}{3}t}}$$

The unique solution of the initial value problem

- Diff Eq ⇒ Infinitely many Solutions
- Initial Value Problem of Diff Eq ⇒ Unique Solution.

Solve $\frac{dy}{dt} = \frac{1}{3}y$.

Method 1 (Integrating Factor)

$\frac{dy}{dt} - \frac{1}{3}y = 0$, $\quad e^{-\frac{1}{3}t}\frac{dy}{dt} - \frac{1}{3}e^{-\frac{1}{3}t}y = 0$,

$\frac{d}{dt}\left(e^{-\frac{1}{3}t}y\right) = 0$, $\quad e^{-\frac{1}{3}t}y = C \Rightarrow \boxed{y = Ce^{\frac{1}{3}t}}$

Method 2 (Separate Variables)

$\frac{dy}{y} = \frac{1}{3}dt$, $\quad \int \frac{dy}{y} = \int \frac{1}{3}dt$,

$\ln|y| = \frac{1}{3}t + C_0$, $\quad |y| = e^{\frac{1}{3}t + C_0} = e^{C_0} \cdot e^{\frac{1}{3}t}$.

$y = \underbrace{\pm e^{C_0}}_{\text{denote by } C} e^{\frac{1}{3}t} \Rightarrow \boxed{y = Ce^{\frac{1}{3}t}}$

Population Growth / Decay

Linear Exp. Model (Malthus, 1798)

<u>Assumption</u> Population y changes exponentially.

i.e. $\frac{dy}{dt} = ky$. $\quad y(t) = y_0 e^{kt}$.

Here, k = the net rate of change per unit of population
$= \begin{pmatrix}\text{the birth rate} \\ \text{per capita}\end{pmatrix} - \begin{pmatrix}\text{the death rate} \\ \text{per capita}\end{pmatrix}$.

<u>Example</u> A culture of bacteria grows exponentially.

<u>Observed</u>: # of bacteria increased 25 % in a hour.

<u>Question</u>: How long does it take for the # of bacteria to double?

<u>Solution</u>: $y(t) = y_0 e^{kt}$.

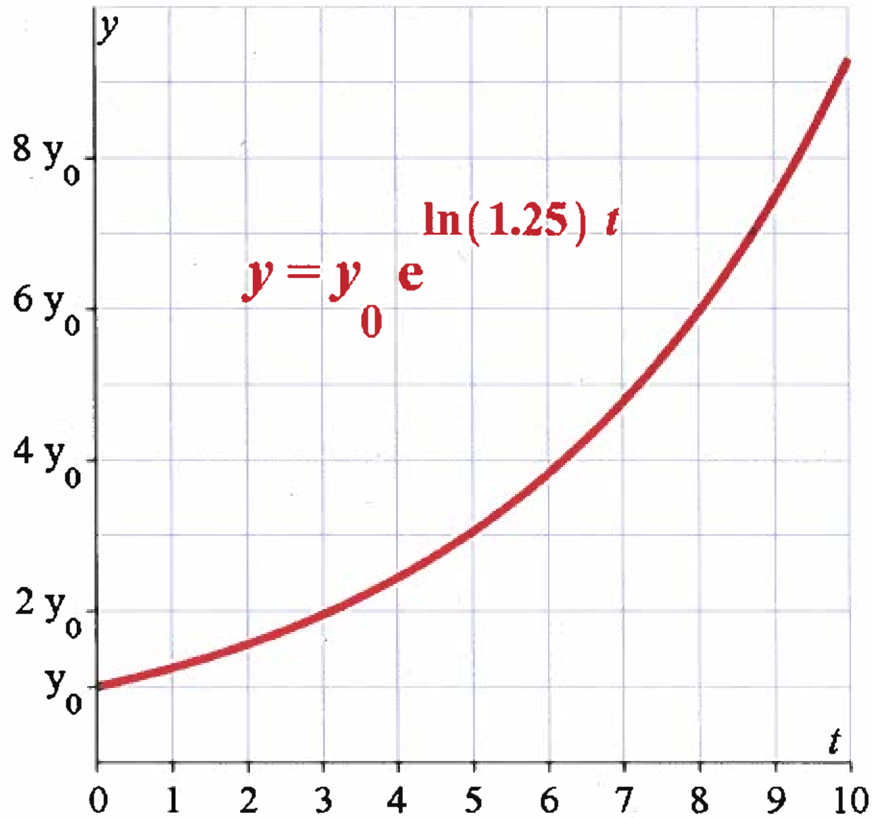
$y(1) = 1.25 y_0 = y_0 e^{k \cdot 1} \Rightarrow 1.25 = e^{k} \Rightarrow k = \ln 1.25$

$y(t) = y_0 e^{(\ln 1.25)t}$.

<u>Q.</u> Find t such that $y(t) = 2y_0$.

$2y_0 = y_0 e^{(\ln 1.25)t} \Rightarrow 2 = e^{(\ln 1.25)t}$

$\Rightarrow \ln 2 = (\ln 1.25)t \Rightarrow t = \frac{\ln 2}{\ln 1.25}$ hours



Answer : Time to Double : $t_2 = \frac{\ln 2}{\ln 1.25} \approx 3.1$ hours

By the way: Time to Triple : $t_3 = \frac{\ln 3}{\ln 1.25} \approx 4.9$ hours

Time to Quadruple: $t_4 = \frac{\ln 4}{\ln 1.25} \approx 6.2$ hours

Time to Octuple : $t_8 = \frac{\ln 8}{\ln 1.25} \approx 9.3$ hours

we always have $t_4 = 2t_2$,
$t_8 = 3t_2$.

Q. Why?

Example

- Coffee Temp. $T(t)$
- Room Temp. 70°F (Surrounding Temp.)
  constant

Newton's Law of Cooling

The rate of decrease of Coffee Temp.
$\propto$ Coffee Temp. − Room Temp.

$$-\frac{dT}{dt} = k(T-70), \qquad T(0) = 190°F.$$

[An initial value problem of diff. eq.]

Solution:

Set $y = T - 70$.

$$-\frac{dy}{dt} = ky. \qquad \begin{cases} \frac{dy}{dt} = -ky \\ y(0) = 190-70 = 120. \end{cases}$$

⇙

Solution: $y(t) = 120e^{-kt}$

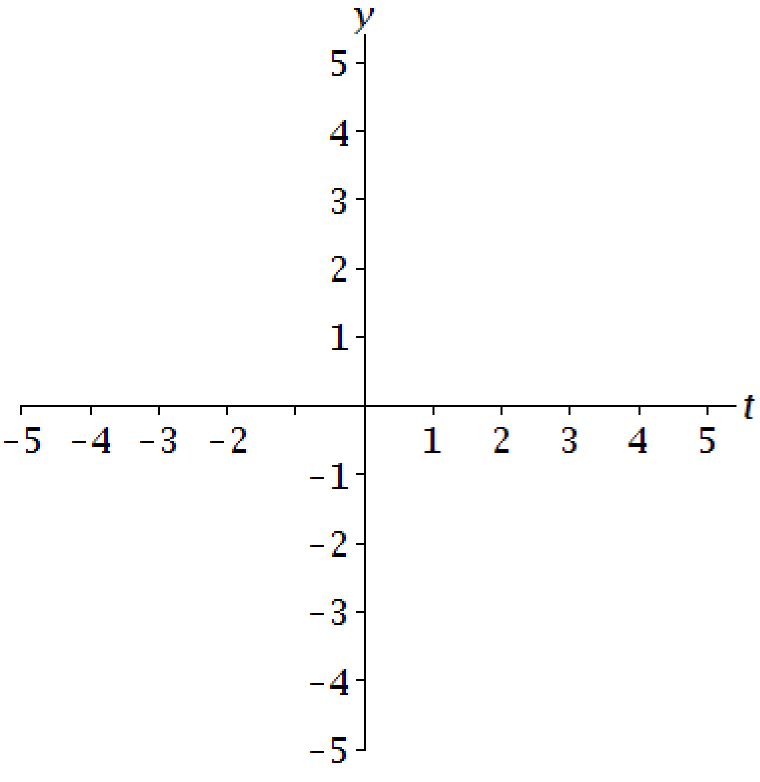
Coffee Temp. $T(t) = 70 + y$

$$= 70 + 120e^{-kt}$$

Direction Fields
Slope Fields

The direction field of

$$\frac{dy}{dt} = \frac{1}{3}y$$



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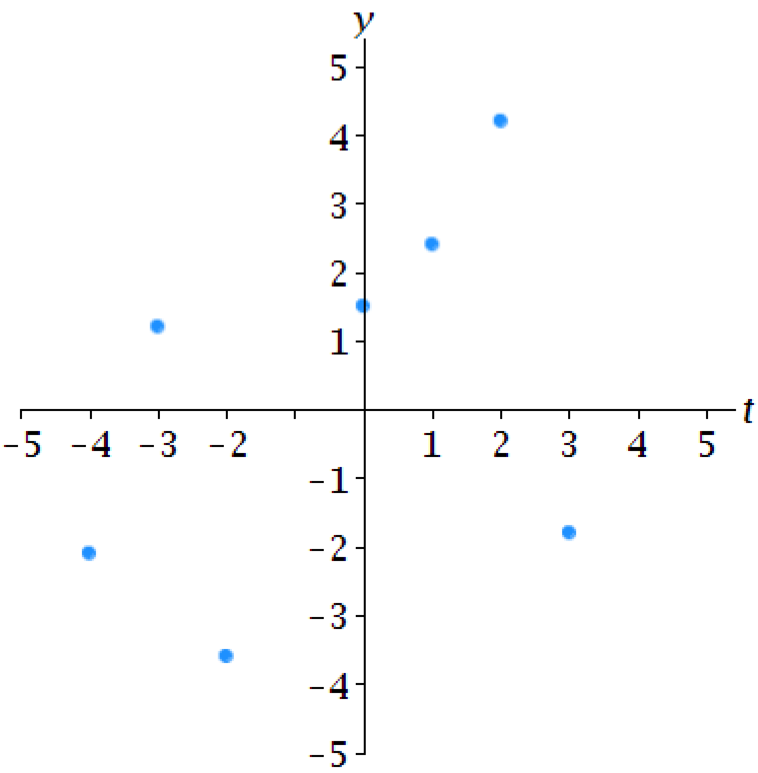

Point (t, y)	Slope $\frac{dy}{dt} = f(t, y) = \frac{1}{3}y$
$(-4, -2.1)$	
$(-3, 1.2)$	
$(-2, -3.6)$	
$(0, 1.5)$	
$(1, 2.4)$	
$(2, 4.2)$	
$(3, -1.8)$	

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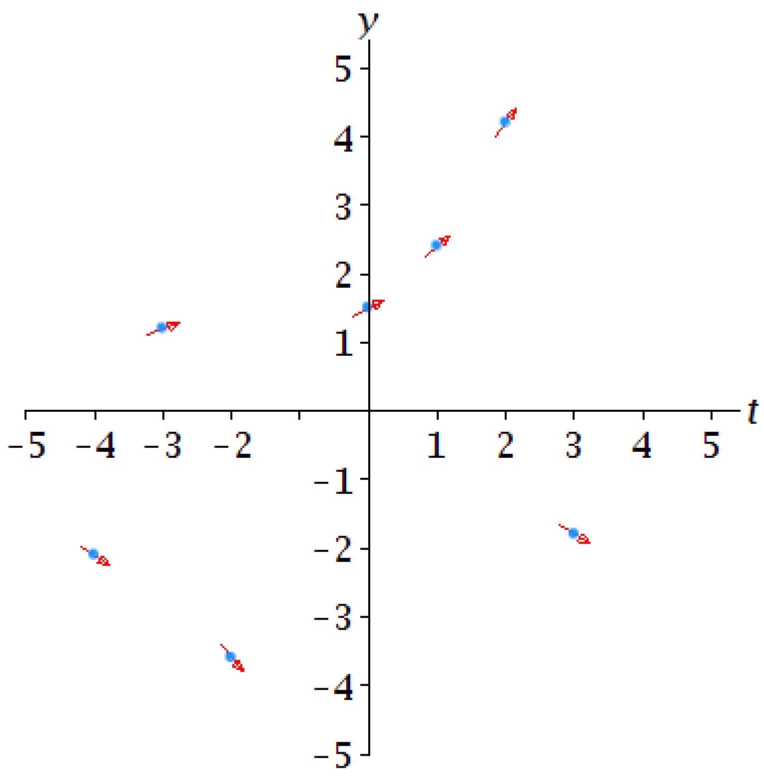
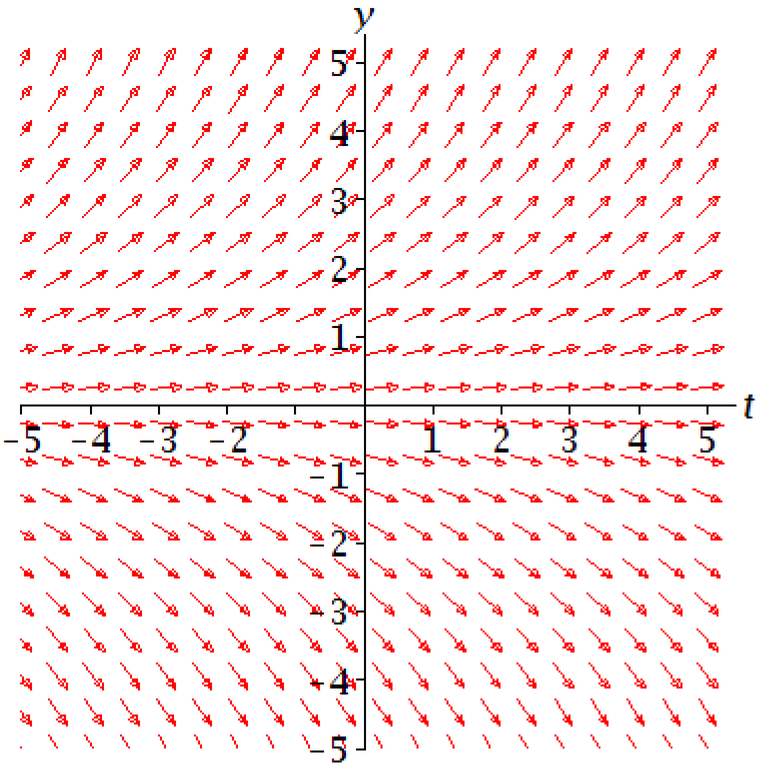


Point (t, y)	Slope $\frac{dy}{dt} = f(t, y) = \frac{1}{3}y$
$(-4, -2.1)$	-0.7
$(-3, 1.2)$	0.4
$(-2, -3.6)$	-1.2
$(0, 1.5)$	0.5
$(1, 2.4)$	0.8
$(2, 4.2)$	1.4
$(3, -1.8)$	-0.6

The direction field of

$$\frac{dy}{dt} = \frac{1}{3}y$$



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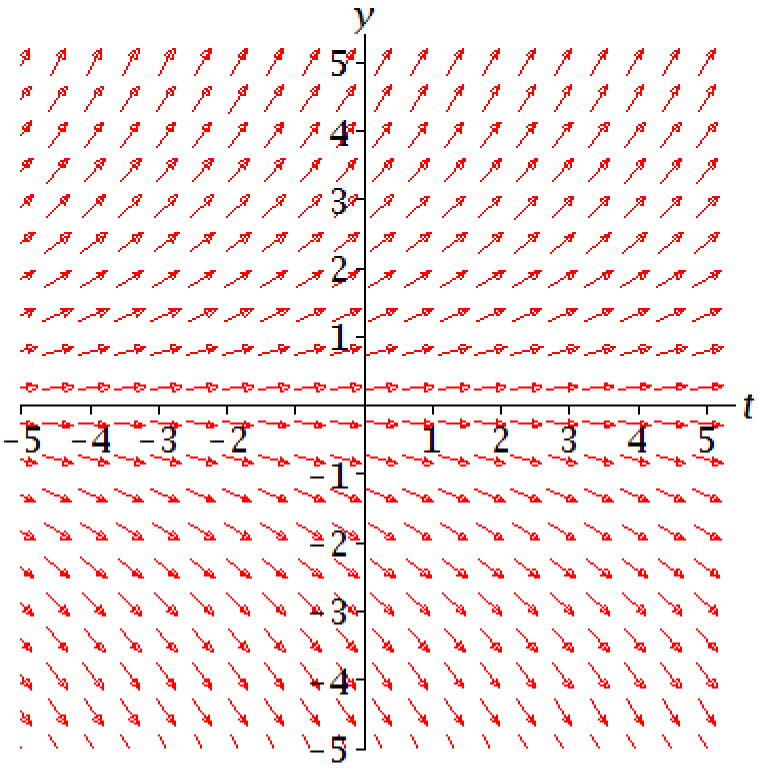
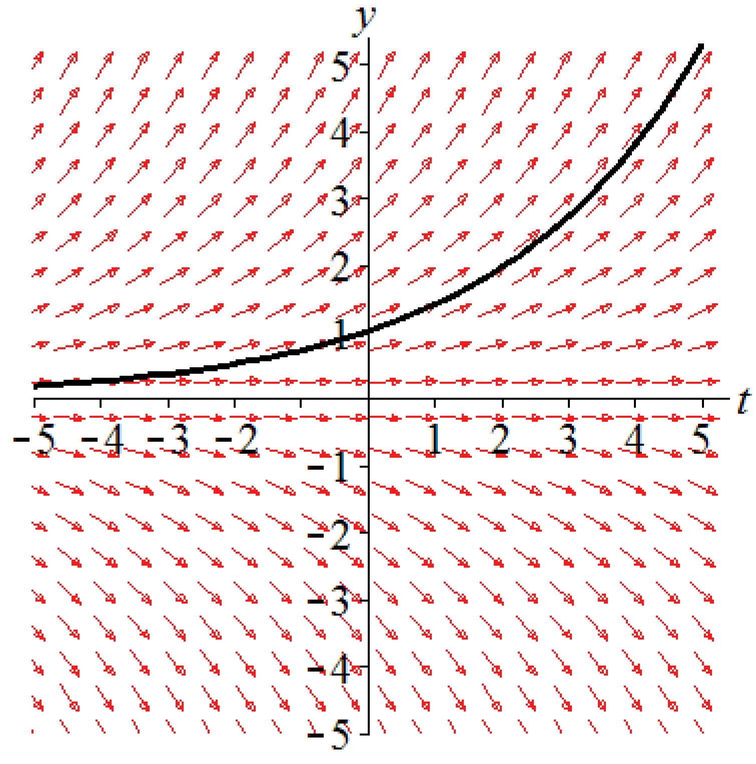


The solution curve of

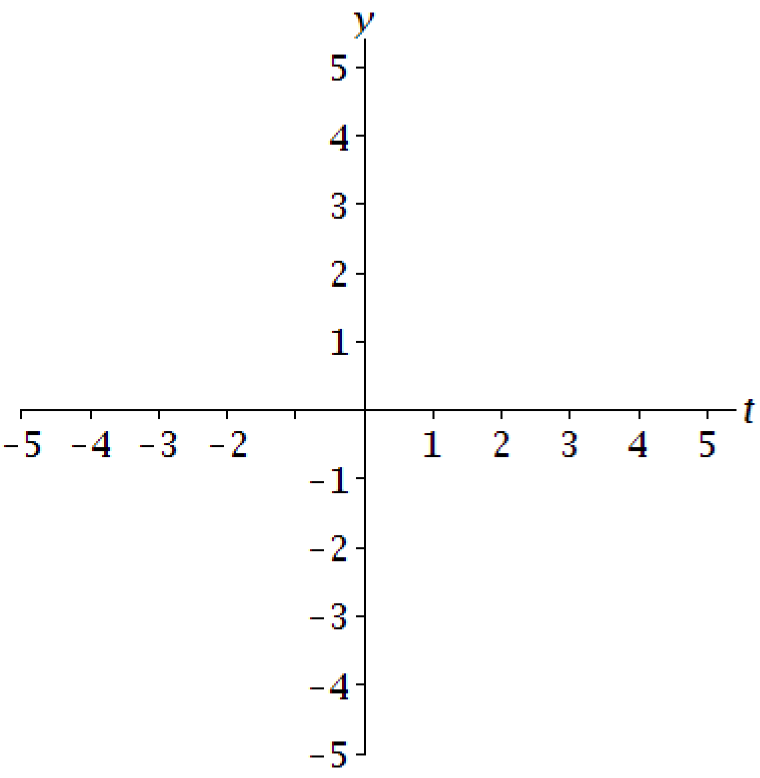
$$\frac{dy}{dt} = \frac{1}{3}y$$

$$y(0) = 1$$



The direction field of

$$\frac{dy}{dt} = \frac{1}{3}y - \frac{y^2}{3 - 3t + t^2}$$



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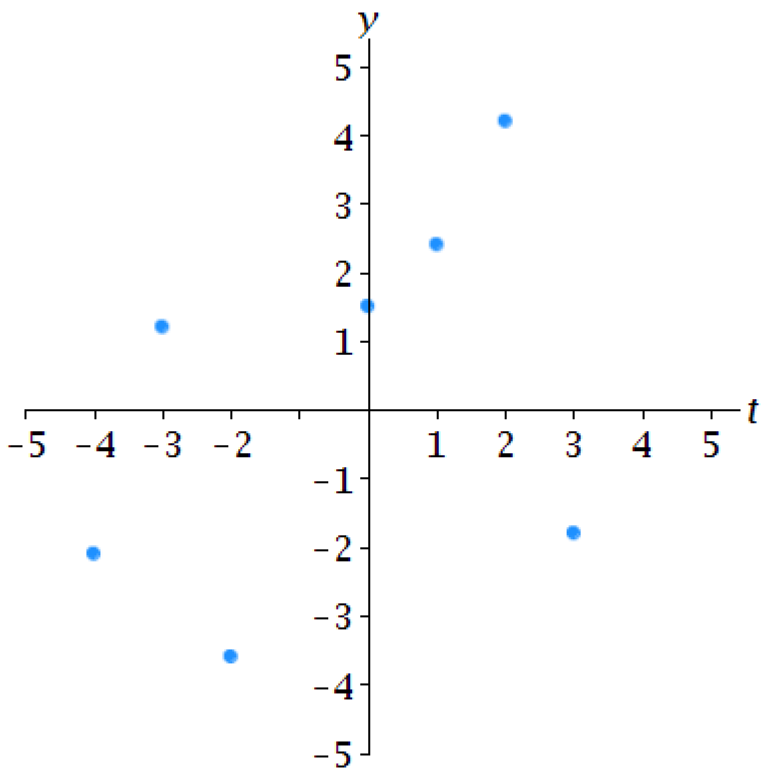

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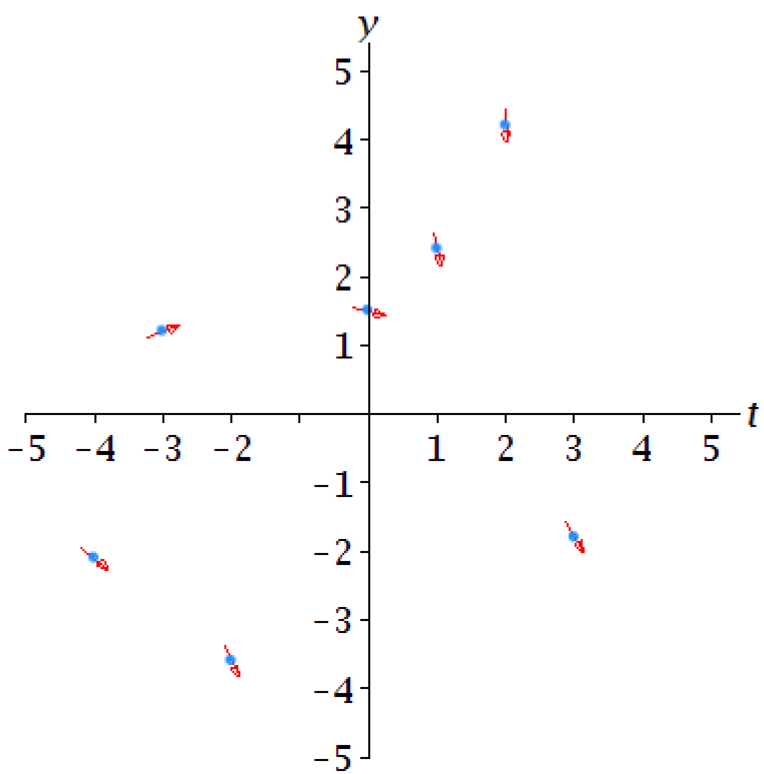
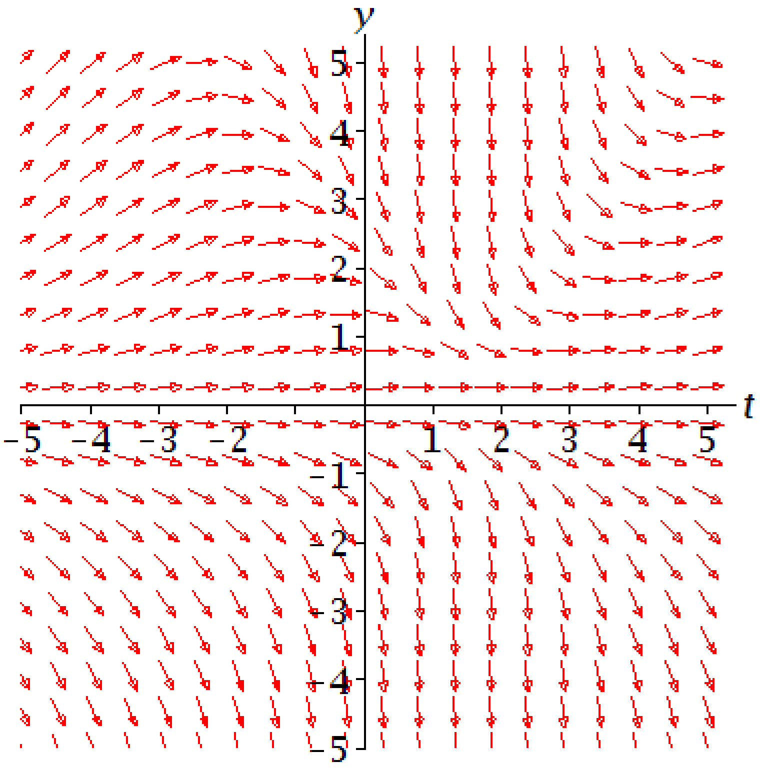


Point (t, y)	Slope $\frac{dy}{dt} = f(t, y)$
$(-4, -2.1)$	$-0.7 - \frac{4.41}{31} = -0.842 \ldots$
$(-3, 1.2)$	$0.4 - \frac{1.44}{21} = 0.331 \ldots$
$(-2, -3.6)$	$-1.2 - \frac{12.96}{13} = -2.1969 \ldots$
$(0, 1.5)$	$0.5 - \frac{2.25}{3} = -0.25$
$(1, 2.4)$	$0.8 - \frac{5.76}{1} = -4.96$
$(2, 4.2)$	$1.4 - \frac{17.64}{1} = -16.24$
$(3, -1.8)$	$-0.6 - \frac{3.24}{3} = -1.68$

The direction field of

$$\frac{dy}{dt} = \frac{1}{3}y - \frac{y^2}{3 - 3t + t^2}$$



The direction field of

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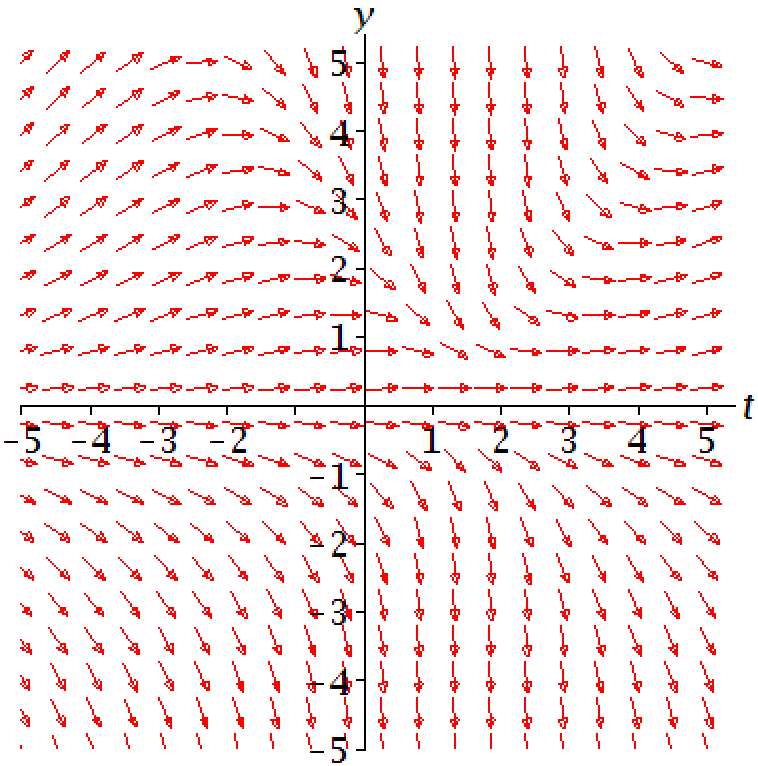

The solution curve of

$$\frac{dy}{dt} = \frac{1}{3}y - \frac{y^2}{3 - 3t + t^2}$$

$$y(0) = 2$$