Coffee Temperature $T(t)$

$T_{room} = 75°F$

$T(0) = 200$

$T(\underset{sec.}{15}) = 195$.

$\boxed{\text{Find } T(t) = ?}$

$\boxed{\text{Find } t \text{ such that } T(t) = 170°F}$

Newton's Cooling Law:

Rate of Coffee Temp Decrease

$\propto$ (Coffee Temp) − (Room Temp.)

$$-T'(t) = k[T(t) - 75]$$

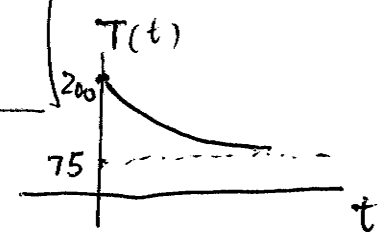
I.V.P
$$\begin{cases} T' = -k(T-75) \\ T(0) = 200 \end{cases}$$

Sol of I.V.P. $T(t) = 75 + 125e^{-kt}$

$$\underset{sec}{T(15) = 195} \Rightarrow 75 + 125e^{-15k} = 195 \Rightarrow k = -\frac{1}{15}\ln\frac{120}{125} = \frac{1}{15}\ln\frac{25}{24}$$

$$\approx 0.00272 \,(sec^{-1})$$

$$\Rightarrow \boxed{T(t) = 75 + 125e^{-\left(\frac{1}{15}\ln\frac{25}{24}\right)t}}$$



· Find t such that $T(t) = 170$:

$$75 + 125e^{-\left(\frac{1}{15}\ln\frac{25}{24}\right)t} = 170$$

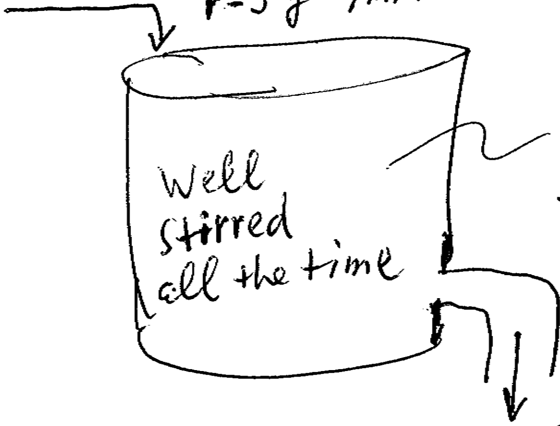
$$-\left(\frac{1}{15}\ln\frac{25}{24}\right)t = \ln\left(\frac{95}{125}\right) = \ln\left(\frac{19}{25}\right)$$

$$t = 15\frac{\ln\left(\frac{25}{19}\right)}{\ln\left(\frac{25}{24}\right)} \approx 100.84 \text{ sec}$$

Salt in a Tank

Inflow Brine $r = 3$ gal/min

Salt concentration $\frac{1}{4}$ lb/gal.



Brine Volume: 100 gal
Initial Amount of Salt: 10 lb. $Q(0) =$

Outflow $r = 3$ gal/min

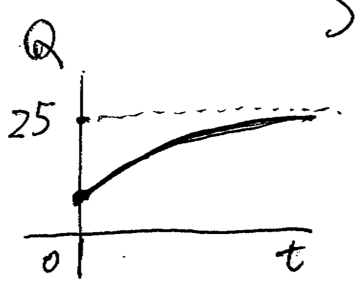
Question Find the amount of salt $Q(t)$

(Rate of Change of Salt) = (Rate of Salt in) − (Rate of Salt Out)

$$Q'(t) = 3 \times \frac{1}{4} - 3 \times \frac{Q(t)}{100}$$

Initial Value Problem
$$\begin{cases} Q' = \frac{3}{4} - \frac{3}{100} Q \\ Q(0) = 10 \end{cases}$$

Sol. $Q(t) = 25 - 15e^{-0.03t}$ (lb). [after t min]



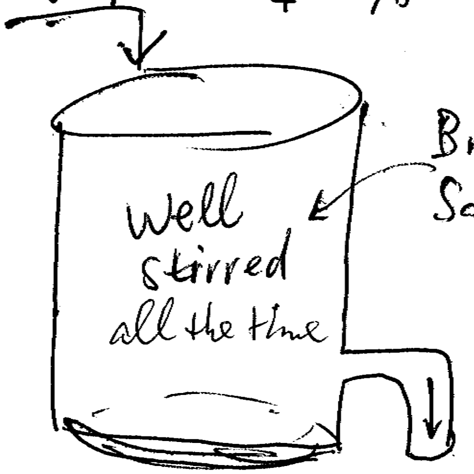
Question 2 Find t such that $Q(t) = 20$ (lb)

$$25 - 15e^{-0.03t} = 20$$

$$\Rightarrow t = \frac{\ln(5/15)}{-0.03} = \frac{100}{3} \ln 3 \approx 36.62 \text{ (min)}$$

$$\approx 36 \text{ min } 37 \text{ sec.}$$

Brine $r = 3$ gal/min Salt Concentration $\frac{1}{4}$ lb/gal



Brine Vol. 100 gal
Salt $Q(0) = 10$ lb

$r = 3$ gal/min

$\boxed{\text{Find } Q(t) = ?}$

$$\begin{cases} Q' = \frac{1}{4} \times 3 - \frac{Q}{100} \times 3 \\ Q(0) = 10 \end{cases}$$

$\Downarrow$

$$Q(t) = 25 - 15e^{-0.03t} \quad (\text{lb})$$

$\boxed{\text{Find } t \text{ such that } Q(t) = 20 \text{ (lb)}}$

$$25 - 15e^{-0.03t} = 20 \Rightarrow t = \frac{\ln(5/15)}{-0.03} = \frac{100 \ln 3}{3}$$

$$\approx 36.62 \text{ (min)} \approx 36 \text{ min } 37 \text{ sec}$$

Modify the Assumption

The Rate of Brine Out $r_{Out} = 2.5$ gal/min

$\Downarrow$

[Total Vol. in Tank after t (min)] $= 100 + 0.5t$ (gal)

$$\begin{cases} Q' = \frac{1}{4} \times 3 - \frac{Q}{100 + 0.5t} \times 2.5 \\ Q(0) = 10 \end{cases} \Rightarrow Q(t) = 25 + \frac{1}{8}t + C(100 + 0.5t)^{-5}$$

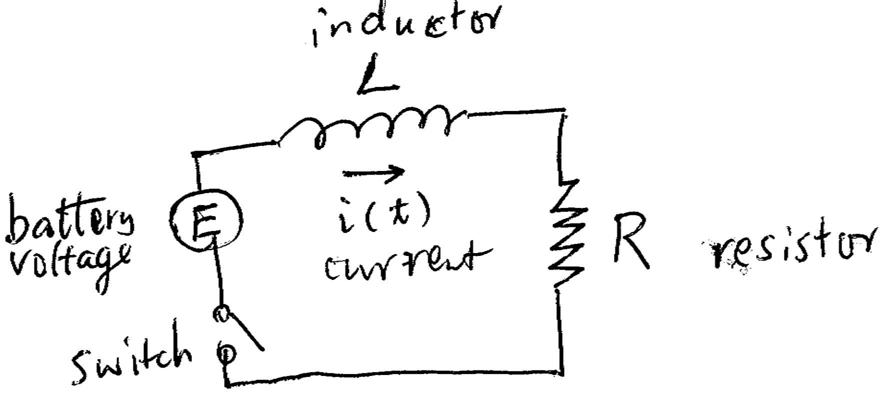
$$Q(0) = 10 \Rightarrow C = -15 \times 100^5$$

$$Q(t) = 25 + \frac{1}{8}t - 15\left(1 + \frac{1}{200}t\right)^{-5}$$

$$= 25 + 0.125t - 15(1 + 0.005t)^{-5} \quad (\text{lb})$$

$\boxed{\text{Find } t \text{ s.t. } Q(t) = 20 \text{ (lb)}}$ $t \approx 25.645$ (min) ≈ 25 (min) 39 (sec)

RL Circuit

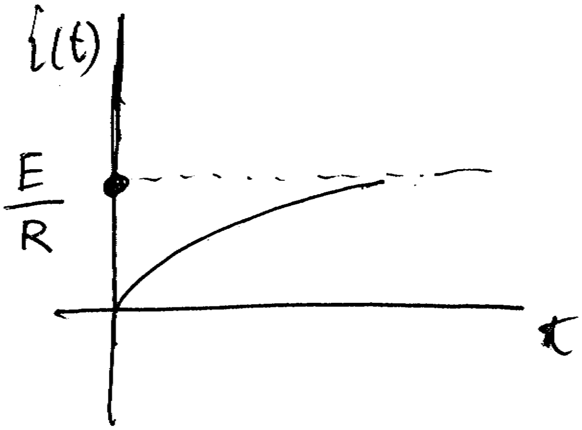


Kirchhoff's Law

$$L \frac{di}{dt} + R i = E$$

Gen. Sol.s $\quad i(t) = \frac{E}{R} + C e^{-\frac{R}{L} t}$

Init. Cond $\; i(0) = 0 \Rightarrow i(t) = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} t}$



Example
$$\begin{cases} E = 12 \text{ volt}, \\ L = \frac{1}{2} \text{ henry} \\ R = 10 \text{ ohms} \end{cases} \quad i(0) = 0.$$

Current $i(t)$

$$\left.\begin{aligned} \frac{1}{2}\frac{di}{dt} + 10 i = 12 \\ i(0) = 0 \end{aligned}\right\} \Rightarrow i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t}$$

- **Replace E by e(t)** $\quad L \frac{di}{dt} + R i = e(t)$

$\Rightarrow$ integrating factor

$$i(t) = e^{-\frac{R}{L}t} \int \frac{e(t)}{L} e^{\frac{R}{L}t} dt + C e^{-\frac{R}{L}t}$$

A Leaking Tank

Torricelli's Principle

Outflow velocity $= \sqrt{2gh}$

Leaking Rate

$=$ Rate of Vol. Decrease

$\propto A\sqrt{2gh}$

$k = 0.6$ for water

$$\frac{dV}{dt} = -kA\sqrt{2gh} \quad \Rightarrow \quad h' = -\frac{kA}{\pi R^2}\sqrt{2g}\sqrt{h}$$

$$\frac{d}{dt}(\pi R^2 h)$$

$$\begin{cases} h' = -r\sqrt{h} \\ h(0) = h_0 \end{cases}$$

$$\int \frac{1}{\sqrt{h}}\, dh = \int -r\, dt$$

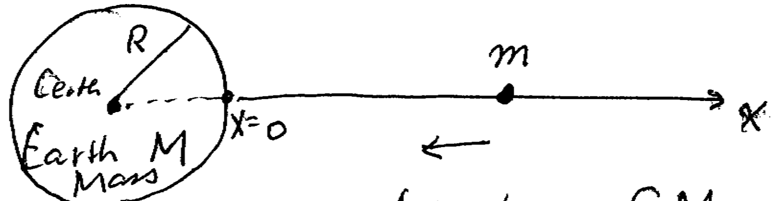
$$2\sqrt{h} = -rt + C$$

$$2\sqrt{h_0} = C$$

$$2\sqrt{h} = -rt + 2\sqrt{h_0}$$

$$\boxed{h = \left(\sqrt{h_0} - \frac{rt}{2}\right)^2}$$

$\boxed{\text{Escape Velocity}}$ Mass m projected from the earth surface initial velocity V_0



Attraction $\dfrac{GMm}{(R+x)^2} = \dfrac{mgR^2}{(R+x)^2}$

- On the earth surface:

$$x=0 \Rightarrow \frac{GMm}{R^2} = mg \Rightarrow GM = gR^2.$$

- Newton's Law

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$$

$$\left\{\begin{array}{l} \dfrac{dv}{dt} = -\dfrac{gR^2}{(R+x)^2} \\ \dfrac{dx}{dt} = v \end{array}\right\} \xRightarrow{\text{divide}} \frac{dv}{dx} = \frac{-\dfrac{gR^2}{(R+x)^2}}{v}$$

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}$$

$$\frac{d}{dx}\left(\frac{v^2}{2}\right)$$

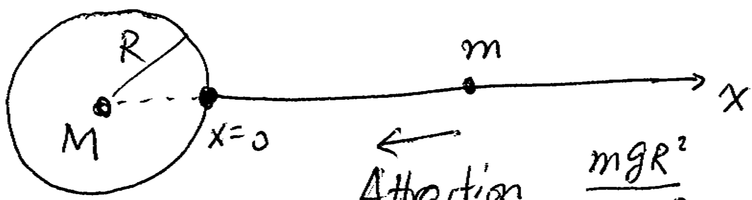
Integrate:

$$\frac{v^2}{2} = \int \frac{-gR^2}{(R+x)^2}\,dx = \frac{gR^2}{R+x} + C.$$

At $t=0$ $\left\{\begin{array}{l} x=0 \\ v=V_0 \end{array}\right. \Rightarrow \dfrac{V_0^2}{2} = \dfrac{gR^2}{R} + C \Rightarrow C = -gR + \frac{1}{2}V_0^2$

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} - gR + \frac{1}{2}V_0^2 \Rightarrow \boxed{\frac{1}{2}v^2 = \frac{-gRx}{R+x} + \frac{1}{2}V_0^2}$$

Escape Velocity (Continued)



Initial Velocity V_0.

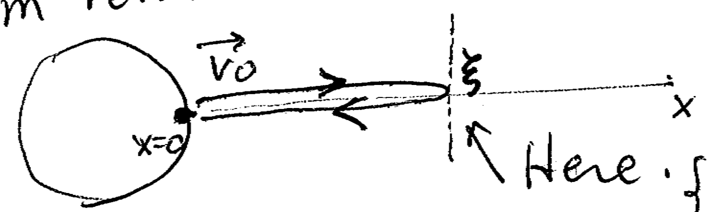
$\frac{mgR^2}{(R+x)^2}$.

$$\boxed{\frac{1}{2}v^2 = -\frac{gRx}{R+x} + \frac{1}{2}V_0^2}$$

Some critical speed.

- When V_0 is not too large, $V_0 < V_e$

m returns to earth



Here. $\begin{cases} x = \text{maximum } \xi \\ v = 0 \end{cases}$

$$0 = -\frac{gR\xi}{R+\xi} + \frac{1}{2}V_0^2$$

$$\Rightarrow V_0 = \sqrt{2gR\frac{\xi}{R+\xi}} < \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

≈ 11200 m/s.

≈ 7 miles/s

≈ 25000 mph

$\approx$ Mach 34 (i.e. 34 × Sound speed)

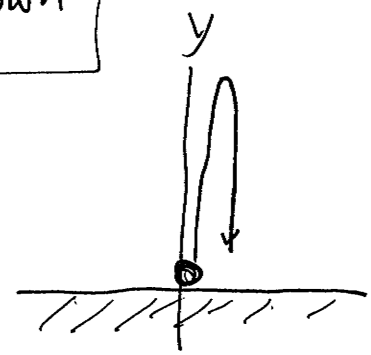
That's escape velocity V_e

- When $V_0 < V_e$, but $V_0 \approx V_e$
  the maximal distance ξ is very lage

- When $V_0 ? V_e$.
  the maximal distance ξ does not exist!

i.e. No Return!

A Ball Thrown Up & Falling Down



Forces $= \begin{cases} \text{Gravity} & mg \\ \text{Air Resistance} & bv^2 \end{cases}$

Newton's Law : $ma = (\text{Net force})$

Up : $my'' = -mg - by'^2$

Down : $my'' = -mg + by'^2$ $\qquad y' = v$

Up : $v' = -g - \frac{b}{m} v^2$

Down : $v' = -g + \frac{b}{m} v^2$

Terminal $\underbrace{\text{Speed}}_{v_T}$ of Free Fall :

$$-g + \frac{b}{m} v_T^2 = 0 \Rightarrow v_T = \sqrt{\frac{mg}{b}}.$$

or $\frac{b}{m} = \frac{g}{v_T^2}$.

Objects	Terminal Speed
Basketball	20 m/s
Baseball	40 m/s
Ping pong	9 m/s
Skydiver (parachute)	5-9 m/s
Skydiver (spread-eagle)	50-60 m/s
Skydiver (dive)	100 m/s
Raindrop	7 m/s .

Up : $v' = -g - \frac{g}{v_T^2} v^2$

Down : $v' = -g + \frac{g}{v_T^2} v^2$

<u>Question</u> Longer to Rise or to Fall ?

(Continued).

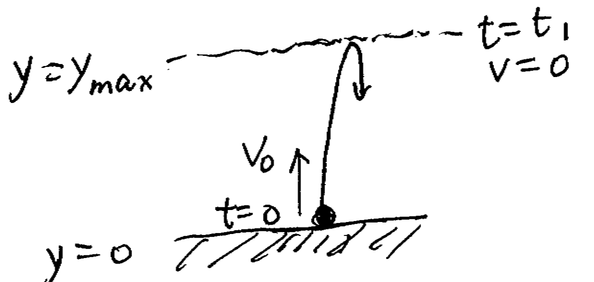
Example Basketball : $V_T = 20\ m/s$.

Initial Velocity $V_0 = 15\ m/s$ (≈ 34 mph)

Gravitational Acceleration $g = 9.8\ m/s^2$

Up. $v' = -g - \frac{g}{V_T^2} v^2$. $\begin{cases} v' = -9.8 - \frac{9.8}{20^2} v^2 \\ v(0) = 15 \end{cases}$

Solve this & find t_1 such that $v(t_1) = 0$.

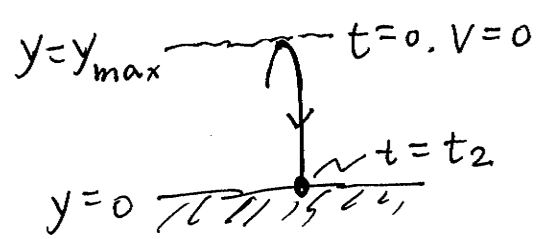


Also find the maximal height

$$y_{max} = \int_0^{t_1} v(t)\,dt$$

$\boxed{\text{Use Computer!}}$

Down $v' = -g + \frac{g}{V_T^2} v^2$ $\begin{cases} v' = -9.8 + \frac{9.8}{20^2} v^2 \\ v(0) = 0 \end{cases}$.



- Solve this to get $v(t)$
- $y(t) = y_{max} + \int_0^t v(\tau)\,d\tau$ Use Computer
- Find t_2 such that $y(t_2) = 0$.

Question Longer to Rise or to Fall?

i.e. $t_1 > t_2$ or $t_1 < t_2$?

Equivalently, you may check the down sol. $y(t)$.

Is $y(t_1) < 0$ or $y(t_1) > 0$?

Use Computer!