Linear Model of Population Dynamics

Malthus (1798)

Assumption

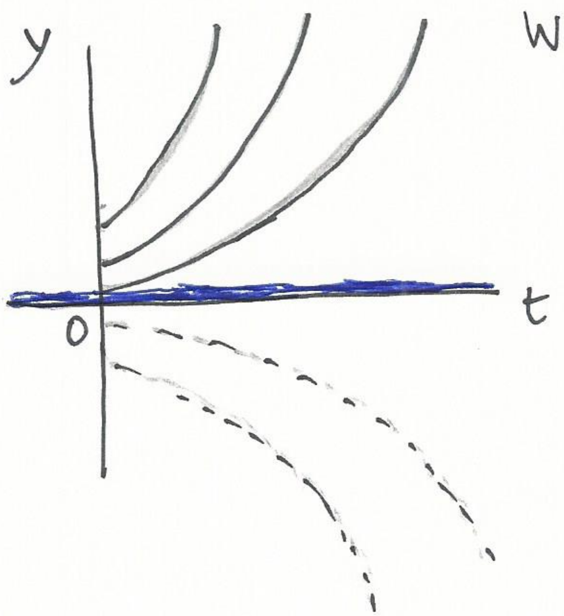
[The growth rate of a population $y(t)$]

= [a constant r]

Diff Eq $\quad \dfrac{dy}{dt} = r\,y$

Init. Cond. $\quad y(0) = y_0$

$\Rightarrow$ Solution: $y(t) = y_0 e^{rt}$.



When $r > 0$: population grows without bound as $t \to \infty$.

- For $y(0) = 0$, we have $y(t) \equiv 0$ for all t.
  (i.e. $y(t) = 0$ is an equilibrium is a time-indep. sol.)
- For $y(0) \approx 0$, do we have $y(t) \approx 0$ for all $t > 0$?

Answer: No.

We say: $y(t) = 0$ is an unstable equilibrium.

Nonlinear Logistic Model of Population Dynamics

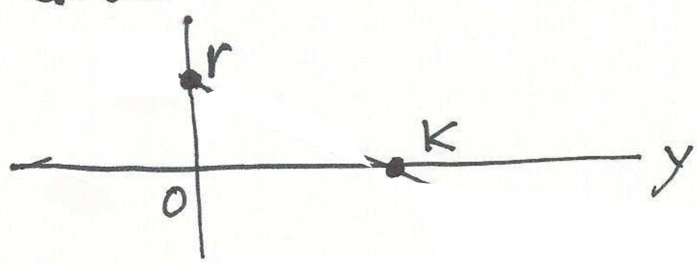
Verhulst (1838)

Assumptions

$$\begin{cases} \bullet \text{ Growth Rate depends on population } y. \\ \bullet\ y \approx 0 \Rightarrow \text{Growth Rate} \approx r\ (>0) \\ \bullet\ y > K \Rightarrow \text{Growth Rate} < 0. \end{cases}$$

Simplest Choice:

Growth Rate vs. y



where $\begin{cases} r = \text{the intrinsic growth rate} \\ K = \text{the carrying capacity} \end{cases}$ (the growth rate when the resource limit is unimportant)

(the upper limit of the population that the resource can support)

Nonlinear Logistic Model of Population Dynamics

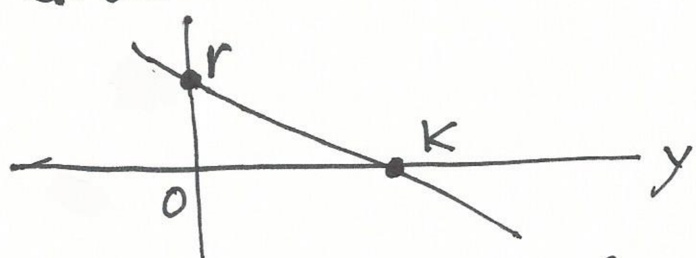
Verhulst (1838)

Assumptions

$$\begin{cases} \cdot\ \text{Growth Rate depends on population } y. \\ \cdot\ y \approx 0 \Rightarrow \text{Growth Rate} \approx r\ (>0) \\ \cdot\ y > K \Rightarrow \text{Growth Rate} < 0. \end{cases}$$

Simplest Choice:

Growth Rate vs. y



i.e. $\text{Growth Rate} = r\left(1 - \frac{y}{K}\right)$.

where $\begin{cases} r = \text{the intrinsic growth rate} \\ K = \text{the carrying capacity} \end{cases}$

r: (the growth rate when the resource limit is unimportant)

K: (the upper limit of the population that the resource can support)

Nonlinear Logistic Model of Population Dynamics

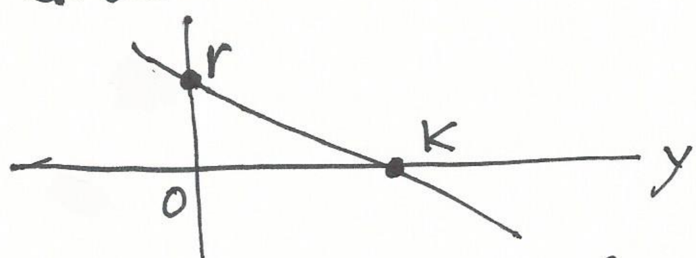
Verhulst (1838)

Assumptions

- Growth Rate depends on population y.
- $y \approx 0 \Rightarrow$ Growth Rate $\approx r \; (>0)$
- $y > K \Rightarrow$ Growth Rate < 0.

Simplest Choice:

Growth Rate vs. y



i.e. Growth Rate $= r\left(1 - \frac{y}{K}\right)$.

$$\begin{cases} \text{Diff Eq.} & \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right) y \\ \text{Init. Cond.} & y(0) = y_0 \end{cases}$$

where r = the intrinsic growth rate (the growth rate when the resource limit is unimportant)

K = the carrying capacity (the upper limit of the population that the resource can support)

Nonlinear Logistic Model of Population Dynamics

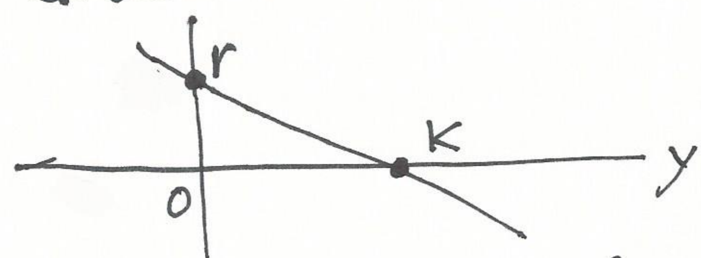
Verhulst (1838)

Assumptions

$$\begin{cases} \bullet \text{ Growth Rate depends on population } y. \\ \bullet\ y \approx 0 \Rightarrow \text{Growth Rate} \approx r\ (>0) \\ \bullet\ y > K \Rightarrow \text{Growth Rate} < 0. \end{cases}$$

Simplest Choice:

Growth Rate vs. y



i.e. Growth Rate $= r\left(1 - \frac{y}{K}\right)$.

$$\left\{\begin{array}{ll} \text{Diff Eq.} & \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right) y \\ \text{Init. Cond.} & y(0) = y_0 \end{array}\right\}$$

where $\begin{cases} r = \text{the intrinsic growth rate (the growth rate when the resource limit is unimportant)} \\ K = \text{the carrying capacity (the upper limit of the population that the resource can support)} \end{cases}$

Solution Formula

$$y(t) = \frac{K y_0}{y_0 + (K - y_0) e^{-rt}}$$

Example

$$\begin{cases} \dfrac{dy}{dt} = 0.03\left(1-\dfrac{y}{4}\right)y \\ y(0) = y_0 \end{cases}$$

Find the Solution $y(t)$.

Solution · The Diff Eq. is a separable eq.

· $\dfrac{dy}{\left(1-\frac{y}{4}\right)y} = 0.03\,dt$.

· Partial Fraction $\dfrac{1}{\left(1-\frac{y}{4}\right)y} = \dfrac{4}{(4-y)y} = \dfrac{1}{4-y} + \dfrac{1}{y}$

· $\int \left[\dfrac{1}{4-y} + \dfrac{1}{y}\right] dy = \int 0.03\,dt$.

$-\ln|4-y| + \ln|y| = 0.03t + C$,

$\ln\left|\dfrac{y}{4-y}\right| = 0.03t + C$, $\left|\dfrac{y}{4-y}\right| = e^{0.03t+C}$

$\dfrac{y}{4-y} = \pm e^{C} \cdot e^{0.03t}$, $\dfrac{4-y}{y} = \pm e^{-C} \cdot e^{-0.03t}$.

$\dfrac{4}{y} - 1 = C_1 e^{-0.03t}$ (where $C_1 = \pm e^{-C}$)

Init. Cond. $y(0) = y_0 \Rightarrow \dfrac{4}{y_0} - 1 = C_1$

$\dfrac{4}{y} - 1 = \left(\dfrac{4}{y_0} - 1\right) e^{-0.03t} = \dfrac{4-y_0}{y_0} e^{-0.03t}$

$\dfrac{4}{y} = 1 + \dfrac{4-y_0}{y_0} e^{-0.03t} = \dfrac{y_0 + (4-y_0)e^{-0.03t}}{y_0}$

$$\boxed{y = \dfrac{4y_0}{y_0 + (4-y_0)e^{-0.03t}}}$$

Example $\dfrac{dy}{dt} = 0.03\left(1 - \dfrac{y}{4}\right) y.$ $\quad (r = 0.03,\ K = 4)$

Solution: $y(t) = \dfrac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Solution Behavior:

Example $\dfrac{dy}{dt} = 0.03\left(1 - \dfrac{y}{4}\right) y.$ $\quad (r = 0.03,\ K = 4)$

Solution: $y(t) = \dfrac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Solution Behavior :

Example $\dfrac{dy}{dt} = 0.03\left(1-\dfrac{y}{4}\right)y.$ $\quad (r=0.03,\ K=4)$

Solution: $y(t) = \dfrac{4y_0}{y_0+(4-y_0)e^{-0.03t}}$

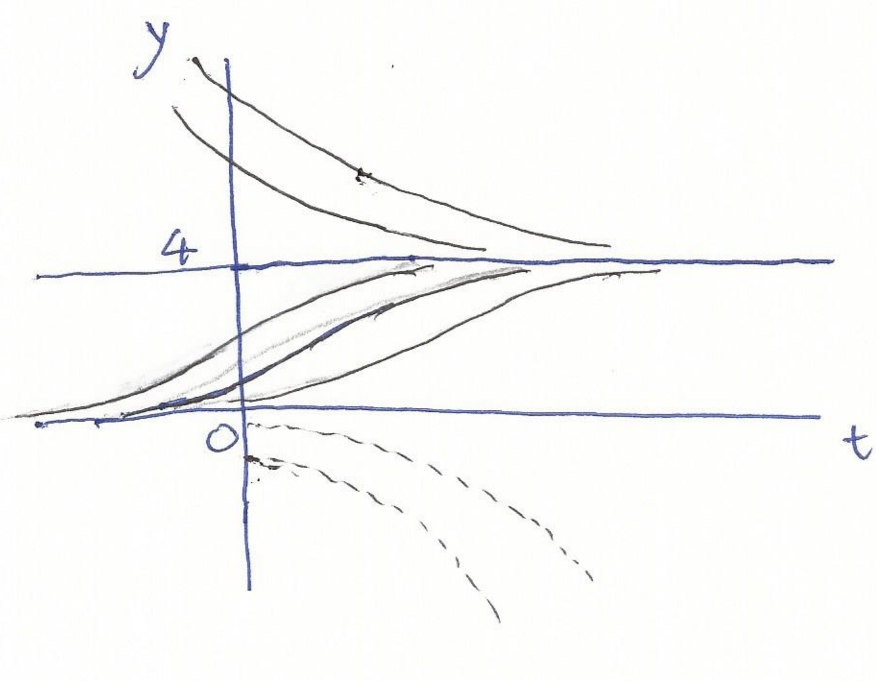
Solution Behavior :

Example $\dfrac{dy}{dt} = 0.03\left(1-\dfrac{y}{4}\right)y.$ $(r=0.03,\ K=4)$

Solution: $y(t) = \dfrac{4y_0}{y_0+(4-y_0)e^{-0.03t}}$

Solution Behavior:

- $y \equiv 0$ is an equilibrium (i.e. a Time-Indep. Sol.)
- $y \equiv 4$ is an equilibrium.
- If $y_0 > 4$, the sol. $y(t)$ decreases.
- If $0<y_0<4$, the sol. $y(t)$ increases.
- All positive solutions $y(t) \to 4$ as $t \to \infty$.



Stability, Asymptotic Stability, & Instability

- For $y(0)=0$, we have $y(t) \equiv 0$ for all t.
  i.e. $y=0$ is an equilibrium (is a time-indep. sol.)

$\boxed{\text{Question}}$:
For $y(0) \approx 0$, do we have $y(t) \approx 0$ for all $t>0$?

$\boxed{\text{Answer}}$ No.

We say: $y=0$ is an unstable equilibrium.

- For $y(0)=4$, we have $y(t) \equiv 4$ for all t.
  i.e. $y=4$ is an equilibrium

$\boxed{\text{Question}}$:
For $y(0) \approx 4$, do we have $y(t) \approx 4$ for all $t>0$?

$\boxed{\text{Answer}}$: Yes.

We say: $y=4$ is a stable equilibrium.

$\boxed{\text{Question}}$:
For $y(0) \approx 4$, do we have $\begin{cases} \text{(i)} \; y(t) \approx 4 \text{ for all } t>0 \,? \\ \text{(ii)} \; \lim_{t\to\infty} y(t) = 4 \,? \end{cases}$

$\boxed{\text{Answer}}$: (i) Yes. (ii) Yes.

We say: $y=4$ is an asymptotically stable equilibrium

Stability, Asymptotic Stability, & Instability

$$\boxed{\frac{dy}{dt} = f(y)}$$

- [$y=b$ is an equilibrium] $\Leftrightarrow$ [$y=b$ is a time-indep. sol.]
  $\Leftrightarrow$ $0 = f(b)$
  $\Leftrightarrow$ [For $y(0)=b$, we have $y(t)=b$ for all t]

- Is $y=b$ an equilibrium? $\Leftrightarrow$ Is $f(b)=0$?

- Is $y=b$ a <u>stable equilibrium</u>?
  $\Leftrightarrow$ For $y(0)\approx b$, do we have $y(t)\approx b$ for all $t>0$?

If Answer is yes, $y=b$ is a <u>stable</u> equilibrium.
If Answer is NO, $y=b$ is an <u>unstable</u> equilibrium.

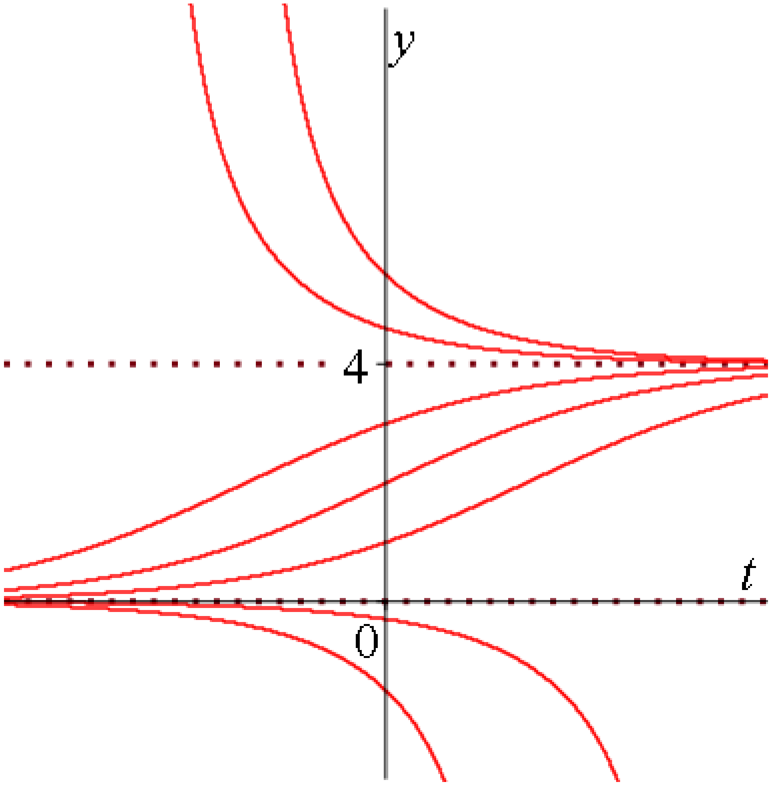
- Is $y=b$ an <u>asymptotically stable equilibrium</u>?
  $\Leftrightarrow$ For $y(0)\approx b$, do we have $\begin{cases} \text{(i) } y(t)\approx b \text{ for all } t>0 \text{ ?} \\ \text{(ii) } \lim_{t\to\infty} y(t) = b \text{ ?} \end{cases}$

If the answer (i) & the answer (ii) are <u>both</u> yes,
$y=b$ is an <u>asymptotically stable</u> equilibrium

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



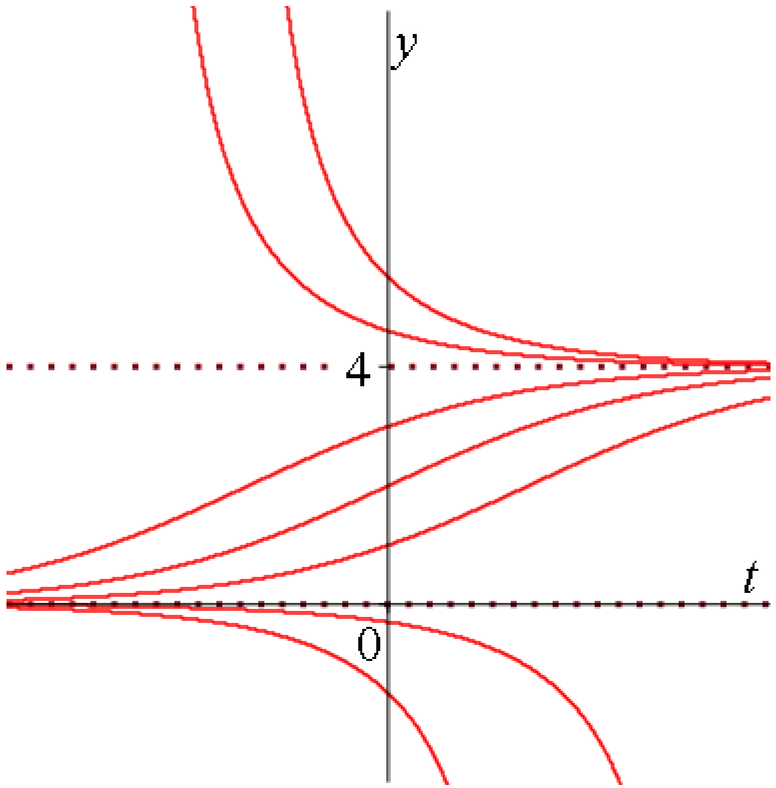
y=0 is an equilibrium. It means:
For $y(0)=0$, we have $y(t)=0$ for all $t>0$.

Question: For $y(0)\approx 0$, do we always have $y(t)\approx 0$ for all $t>0$?

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



y=0 is an equilibrium. It means:
For $y(0)=0$, we have $y(t)=0$ for all $t>0$.

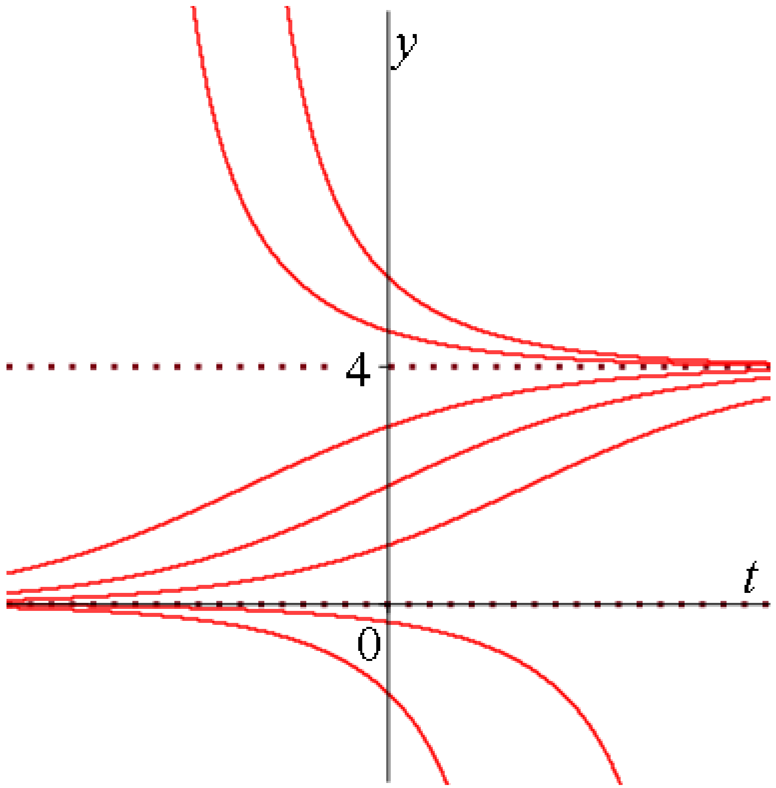
Question: For $y(0)\approx 0$, do we always have $y(t)\approx 0$ for all $t>0$?

Answer: No. **We say:** y=0 is *unstable*.

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



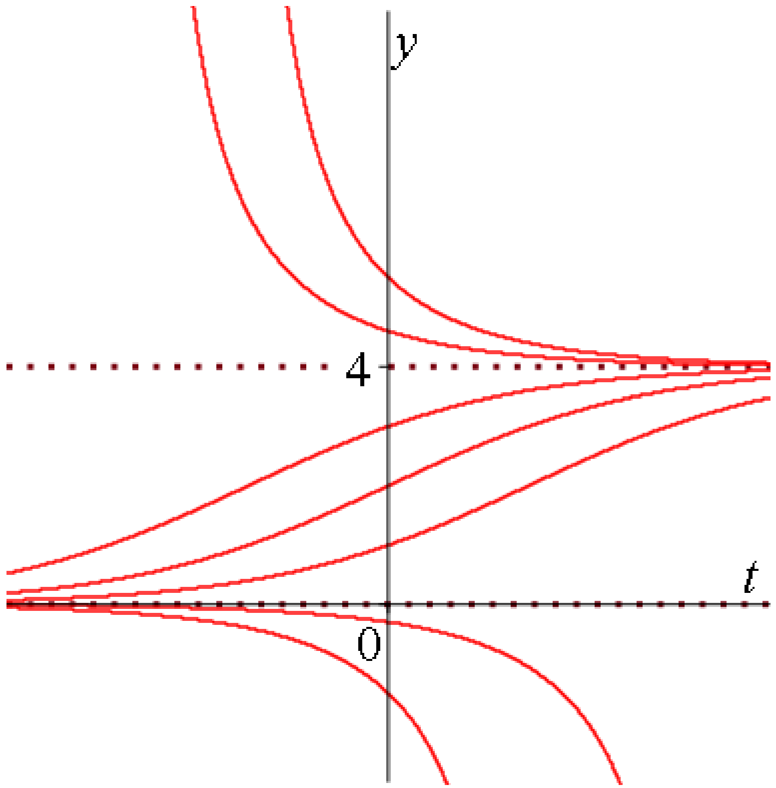
y=4 is an equilibrium. It means:
For *y*(0)=4, we have *y*(*t*)=4 for all *t*>*0.*

Question: For *y*(0)≈4, do we always have *y*(*t*)≈4 for all *t*>0?

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



y=4 is an equilibrium. It means:
For *y*(0)=4, we have *y*(*t*)=4 for all *t*>0.

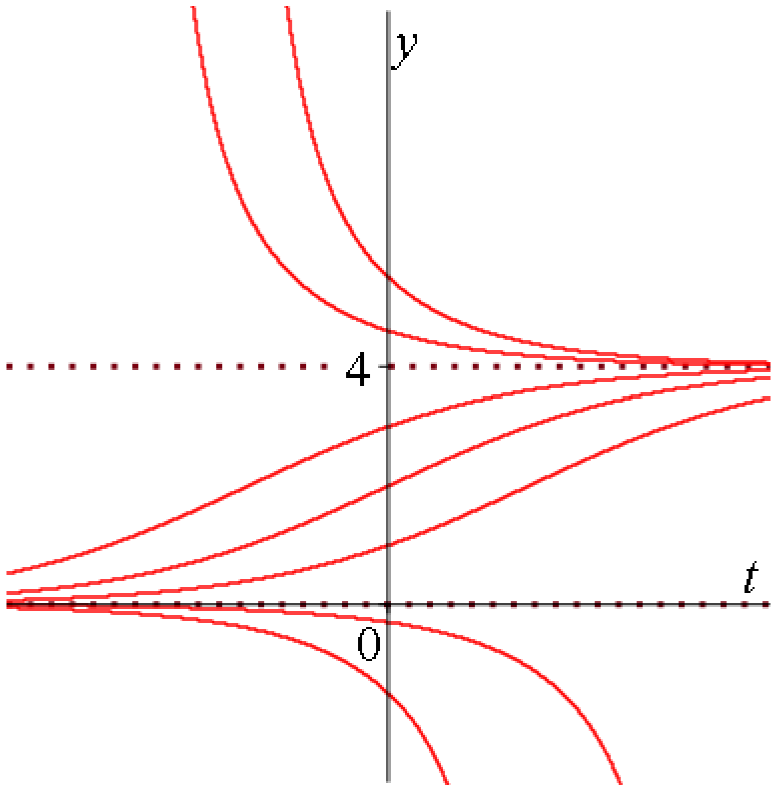
Question: For *y*(0)≈4, do we always have *y*(*t*)≈4 for all *t*>0?

Answer: Yes. **We say:** *y*=4 is *stable*.

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



y=4 is an equilibrium. It means:
For *y*(0)=4, we have *y*(*t*)=4 for all *t>0.*

Question: For *y*(0)≈4, do we always have *y*(*t*)≈4 for all *t*>0?

Answer: Yes. **We say:** *y*=4 is *stable*.

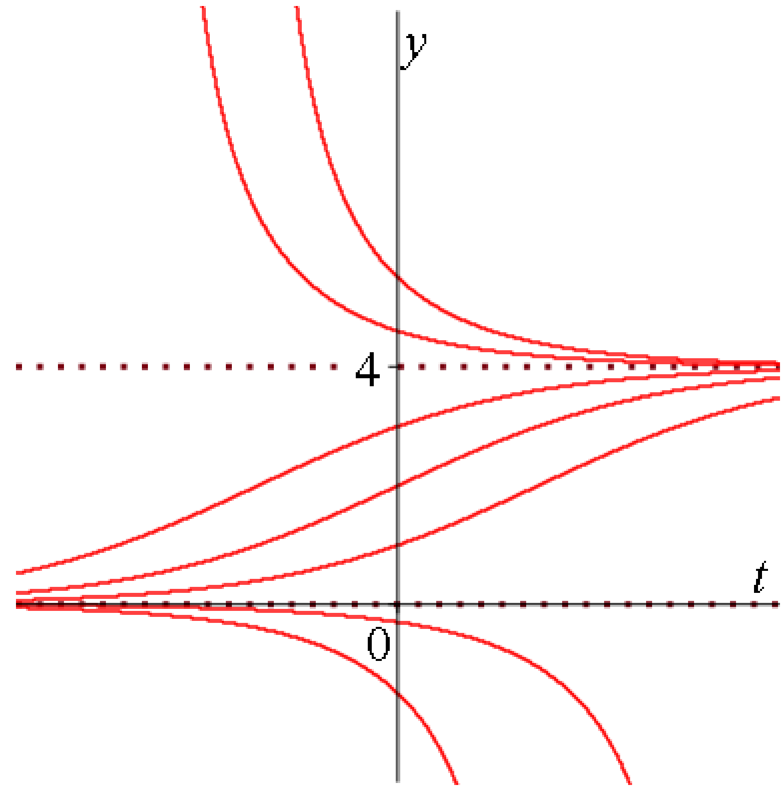
For a stable equilibrium, we ask

A Further Question: For *y*(0)≈4, do we always have $\lim_{t\to\infty} y(t) = 4$?

Diff Eq: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$

General Sol's: $y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$

Equilibria: $y = 0, y = 4$



y=4 is an equilibrium. It means:
For $y(0)=4$, we have $y(t)=4$ for all $t>0$.

Question: For $y(0)\approx 4$, do we always have $y(t)\approx 4$ for all $t>0$?

Answer: Yes. **We say:** y=4 is *stable.*

For a stable equilibrium, we ask

A Further Question: For $y(0)\approx 4$, do we always have $\lim_{t\to\infty} y(t) = 4$?

Answer: Yes. **We say:** y=4 is *asymptotically stable.*